

# Dynamics of Non-Abelian Vortices

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## Abstract

The scattering is studied using moduli space metric for well-separated vortices of non-Abelian vortices in (2+1)-dimensional  $U(N)$  gauge theories with  $N$  Higgs fields in the fundamental representation. Unlike vortices in the Abelian-Higgs model, dynamics of non-Abelian vortices has a lot of new features; The kinetic energy in real space can be transferred to that of internal orientational moduli and vice versa, the energy and charge transfer between two vortices, the scattering angle of collisions with a fixed impact parameter depends on the internal orientations, and some resonances appear due to synchronization of the orientations. Scattering of dyonic non-Abelian vortices in a mass deformed theory is also studied. We find a bound state of two vortices moving along coils around a circle, like a loop of a phone code.

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# 1 Introduction

Topological solitons are localized finite energy solutions of classical equations of motion in field theories. Their stability is protected by topological winding numbers. Various topological solitons are known so far in field theories, for instance, instantons, magnetic monopoles, vortices and kinks (domain walls) [1]. They are not only the classical solutions of the equations of motion but also play important roles in non-perturbative quantum effects through, for example, strong-weak dualities such as the electromagnetic dual in four dimensions. While topological solitons have been observed in condensed matter systems, in cosmology, solitons, especially cosmic strings, may be observed in a future by a direct detection of gravitational waves, the gravitational lens or the cosmic microwave background.

In each dimension, we can consider dynamics of various topological solitons which behave as if they are particle-like objects ; kinks in  $d = 1 + 1$ , vortices in  $d = 2 + 1$ , magnetic monopoles in  $d = 3 + 1$ , and instantons in  $d = 4 + 1$ . In principle, the motion of solitons are determined by field equations which are usually highly nonlinear differential equations. Therefore a very complicated analysis is required to understand the soliton dynamics in general. For complicated physical systems, it is important to extract essential degrees of freedom by throwing away other unimportant degrees of freedom. This procedure is a challenging and interesting problem. Among various kinds of solitons, gauge theories often admit an important class of solitons, called *local solitons*; Yang-Mills instantons [2], 't Hooft-Polyakov monopoles [3], Abrikosov-Nielsen-Olesen (ANO) vortices [4] and  $\mathbb{CP}^1$  kinks (domain walls) [5, 6]. A nice way of extracting such essential degrees of freedom has been established for a particularly important class of local solitons, called Bogomol'nyi-Prasad-Sommerfield (BPS) solitons [7]. BPS solitons saturate the lower energy bound and are the most stable among configurations with a fixed topology [7]. They further naturally appear in supersymmetric gauge theories, break/preserve a fraction of supersymmetry, and consequently are quantum mechanically stable under perturbative or non-perturbative quantum corrections [8]. Since static BPS solitons exert exactly no forces among them at any distance, multiple solitons can statically coexist at any position, which become parameters of multi-soliton solutions, namely moduli parameters associated with massless modes of solitons. Even though no static forces exist, solitons feel forces depending on their velocity. As long as solitons move slowly, it is sufficient to consider the motion of massless modes to describe the dynamics neglecting all the massive modes. This approximation is called the moduli space (geodesic, or Manton) approximation, which was proposed by Manton to discuss the dynamics of BPS monopoles [9, 1]. The moduli space dynamics is a good approximation when the kinetic energy of solitons is much smaller than any mass scales of the theory. According to the moduli space dynamics, solitons move along geodesics of the moduli space of the soliton, where the forces depending on velocities

of solitons are represented as geodesic forces.

Unfortunately, it is not very easy to get the moduli space metric in general. Only in few cases, the metrics are known explicitly; the prime example is the Atiyah-Hitchin metric of  $k = 2$  BPS monopoles in the  $SU(2)$  gauge theory. For multiple monopoles, only asymptotic metrics were obtained when monopoles are well separated [11]. The moduli spaces of well-separated monopoles were also obtained in gauge theories with arbitrary gauge groups [12]. The moduli space approximation has been successfully applied to many other solitons such as ANO vortices in the BPS limit [13, 14, 15, 16, 17, 18], lumps [19], and BPS domain walls [6]. It has been applied even to BPS composite solitons of different kinds [20, 21], such as domain wall networks (webs) [22, 23, 24] and vortex-strings stretched between parallel domain walls [25, 23, 24]. However only the asymptotic metric is explicitly known for well-separated ANO vortices [16, 17, 18], because the ANO vortex equations are not integrable.<sup>1</sup> These metrics have been used to discuss the scattering problems. In particular, scattering processes of solitons have attracted attention of many mathematicians and physicists. It is well known that the magnetic monopoles scatter, surprisingly, with 90 degree, when they collide head-on [10]. The same has been seen for the head-on collision of ANO vortices in the BPS limit [14, 15], and vortex-strings stretched between parallel domain walls [21].

In general the moduli space of BPS solitons is a surprisingly big space. Usually, the dimensions of the moduli space is proportional to the topological number (the number of the solitons)  $k$ . For example, the ANO vortices in the Abelian-Higgs model (Ginzburg-Landau model) at the critical coupling have  $2k$  moduli parameters, which correspond to positions of the vortices [28]. This is much larger than the dimension of the symmetry group of the theory. Some topological solitons admit more moduli than the degrees of freedom of their positions. For example, the  $k$  BPS monopoles in the  $SU(2)$  gauge theory is known to have  $4k = (3 + 1)k$  degrees of freedom. As in the vortex case,  $3k$  can be identified as the monopole positions. The remaining  $k$  degrees of freedom are  $U(1)$  phases of the internal space, which can be called as internal orientations. There are other examples of solitons which possess the  $U(1)$  orientational moduli; kinks in the  $\mathbb{CP}^1$  model [5] and  $U(1)$  or  $U(N)$  gauge theories coupled to Higgs fields with non-degenerated masses [6, 29]. Hence these solitons can be called *Abelian solitons*. On the other hand, non-Abelian moduli are associated with *non-Abelian solitons*; Yang-Mills instantons, non-Abelian monopoles [30], non-Abelian vortices [31], and non-Abelian kinks (in  $U(N)$  gauge theory coupled to Higgs fields with degenerated masses) [32]. The internal orientations can be regarded as Nambu-Goldstone zero modes corresponding to global symmetries or (global parts of) local symmetries,

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<sup>1</sup> For ANO vortices in a hyperbolic plane with a particular curvature, the vortex equations become integrable [26] and consequently the moduli space metric can be calculated [27].

which are unbroken in the vacuum but are spontaneously broken in the presence of solitons. In general, the motion of internal  $U(1)$  orientations give preserved charges to the solitons, making them BPS dyonic solitons.

Since the discovery of non-Abelian vortices [31], much progress has been made in recent years [33, 23, 24]. Unlike the ANO vortices in the Abelian-Higgs model, the non-Abelian vortices have non-Abelian internal orientations and associated conserved charges; In the case of  $U(N)$  gauge theory with  $N$  flavors of Higgs fields in the fundamental representation, the internal orientation is the complex projective space  $\mathbb{C}P^{N-1}$ , which corresponds to Nambu-Goldstone modes associated with the  $SU(N)_{\text{C+F}}$  color-flavor locked global symmetry spontaneously broken in the presence of vortices. Because of non-Abelian internal orientations, we can expect that the dynamics of the non-Abelian vortices is much richer and more interesting compared to the ANO vortices, although the analysis gets much more complicated. The moduli space of multiple vortices with full moduli parameters was completely determined without metric by partially solving BPS vortex equations [25, 34, 24, 35]; The moduli space for  $k$  separated vortices is a  $k$ -symmetric product

$$\mathcal{M}_k^{\text{sep}} \simeq (\mathbb{C} \times \mathbb{C}P^{N-1})^k / \mathcal{S}_k \subset \mathcal{M}_k \quad (1.1)$$

of the single vortex moduli space [34] while the whole space  $\mathcal{M}_k$  is regular. General formula for the moduli space metric and its Kähler potential were given in [36]. The metric of the moduli subspace for two coincident vortices [39, 38, 37], which is supplement to  $\mathcal{M}_{k=2}^{\text{sep}}$  inside the whole space  $\mathcal{M}_{k=2}$ , was found, and it surprisingly shows that two non-Abelian vortices scatter with 90 degree in head-on collision even though they have different internal orientations  $\mathbb{C}P^{N-1}$  as the initial conditions [39]. Most recently, we have obtained the asymptotic metric on the moduli space  $\mathcal{M}_k^{\text{sep}}$  of  $k$  well-separated non-Abelian vortices which is valid when the separation of vortices are much larger than the inverse Compton wave length of massive vector bosons, which is the length scale of the vortices [40].

In this paper, we study the dynamics of non-Abelian vortices by using the recently found asymptotic metric of non-Abelian vortices in the  $U(N)$  gauge theory with  $N$  Higgs scalar fields in the fundamental representation. In order to solve the dynamics, we will make use of the technique of the moduli space approximation. The asymptotic metric of the moduli space allows us to solve the dynamics of the well-separated and slowly moving non-Abelian vortices. We find that the dynamics of the non-Abelian vortex is quite different from that of the Abelian one. The major reason of the difference can be traced back to the conserved charges which are absent in the Abelian case. The vortices with the same charges repel while those with the opposite charges attract. Since the charges can change during the scattering process, non-Abelian vortices experience rich and subtle forces, which produce sometimes the counter-intuitive or unexpected dynamics. We find several new features of the dynamics of non-Abelian vortices:

i) the scattering angle depends on the internal orientation, especially parallel orientations give repulsion while anti-parallel orientations give attraction, ii) the energy of real and internal spaces can be transferred, iii) the energy and charge transfer between two vortices occur, and iv) some resonances appear due to synchronization of the orientations.

We also study the dynamics of the dyonic non-Abelian vortices in the mass deformed theory [41], with the method of the moduli space dynamics. A new feature in this case is that a potential term appears in the low energy effective action. Therefore, the solitons experience two forces: The one is the geometric force and the other is the potential force. In a special situation, the two dyonic vortices drift on a circular orbit. This fact strongly suggests that the non-Abelian dyonic vortices can have a bound state.

The paper is organized as follows. In Section 2 we introduce the model which allows the BPS non-Abelian vortices and review briefly the low energy effective theory of the two non-Abelian vortices, and the moduli space dynamics. We also give the asymptotic metric on the moduli space and the geodesic equations for the well-separated vortices. In Sec. 3, we study the scattering of two non-Abelian vortices. After defining the Noether charges of vortices, we give typical examples of numerical solutions of geodesic motion on the moduli space in Subsec. 3.2. We then analytically study properties of dynamics; the geodesic force in Subsec. 3.3, scattering of two vortices with a large impact parameter by free motion approximation in Subsec. 3.4, and dynamics with zero impact parameter in Subsec. 3.5. In Sec. 4, we consider a mass deformation of the theory and investigate the dynamics of the dyonic non-Abelian vortices. We find a bound state of two dyonic vortices moving along coils around a circle. Sec. 5 is devoted for conclusion and discussion. In Appendix A, the effective action of vortices is written in the  $U(2)$  case in terms of unit three-component vectors.

## 2 Asymptotic metric for non-Abelian vortices

### 2.1 Lagrangian and BPS equations

We consider a  $U(N)$  gauge theory in  $(2+1)$ -dimensional spacetime with gauge fields  $w_\mu$  for  $U(1)$  and  $W_\mu^a$  ( $a = 1, \dots, N^2 - 1$ ) for  $SU(N)_C$ , which couple to  $N$  Higgs fields  $H^A$  ( $A = 1, \dots, N$ ) in the fundamental representation of the  $SU(N)_C$  gauge group. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4e^2}(f_{\mu\nu})^2 - \frac{1}{4g^2}(F_{\mu\nu}^a)^2 + (\mathcal{D}^\mu H^A)^\dagger \mathcal{D}_\mu H^A - V, \quad (2.1)$$

$$V = \frac{e^2}{2}(H_A^\dagger t^0 H^A - \xi)^2 + \frac{g^2}{2}(H_A^\dagger t^a H^A)^2, \quad (2.2)$$

where  $\xi$  is the Fayet-Iliopoulos parameter,  $e$  and  $g$  are gauge coupling constants for  $U(1)$  and  $SU(N)_C$ , respectively. The overall scalar coupling constants in the potential  $V$  are chosen to be equal to the square of the gauge coupling constants, so that the model admits the BPS non-Abelian vortices. Thus the model has the three coupling constants  $e, g, \xi$ . In the three dimensional spacetime, all of the mass dimensions of  $e^2, g^2, \xi$  are unity. Our convention is  $\mathcal{D}_\mu H^A = (\partial_\mu + iw_\mu t^0 + iW_\mu^a t^a)H^A$  and  $f_{\mu\nu}t^0 + F_{\mu\nu}^a t^a = -i[\mathcal{D}_\mu, \mathcal{D}_\nu]$ . The matrices  $t^0$  and  $t^a$  are the generators of  $U(1)$  and  $SU(N)_C$ , normalized as

$$t^0 = \frac{1}{\sqrt{2N}}\mathbf{1}_N, \quad \text{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}. \quad (2.3)$$

As is well known, the Lagrangian Eq. (2.1) can be embedded into a supersymmetric theory with eight supercharges. The Higgs fields can also be expressed as an  $N$ -by- $N$  matrix on which the  $SU(N)_C$  gauge transformations act from the left and the  $SU(N)_F$  flavor symmetry acts from the right

$$H \rightarrow U_C H U_F^\dagger, \quad U_C \in SU(N)_C, \quad U_F \in SU(N)_F. \quad (2.4)$$

The vacuum of this model ( $V = 0$ ) is in an  $SU(N)_{C+F}$  color-flavor locking phase ( $U_F = U_C$ ), where the vacuum expectation values (VEVs) of the Higgs fields are

$$H = \sqrt{c}\mathbf{1}_N, \quad c \equiv \left(\frac{2}{N}\right)^{1/2} \xi. \quad (2.5)$$

In this vacuum, we have a mass gap with the mass  $m_e$  for singlet and  $m_g$  for adjoint representations of  $SU(N)_{C+F}$

$$m_e = e\sqrt{c}, \quad m_g = g\sqrt{c}. \quad (2.6)$$

The energy density for a static configuration can be rewritten as

$$\begin{aligned} \mathcal{E} = & \frac{1}{2e^2} \text{Tr} \left[ f_{12} t^0 - e^2 (H_A^\dagger t^0 H^A - \xi) \right]^2 + \frac{1}{2g^2} \text{Tr} \left[ F_{12}^a t^a - g^2 (H_A^\dagger t^a H^A) \right]^2 \\ & + 4|\mathcal{D}_{\bar{z}} H^A|^2 - \xi \text{Tr}[f_{12} t^0] - i\epsilon^{ij} \partial_i (H_A^\dagger \mathcal{D}_j H^A). \end{aligned} \quad (2.7)$$

For configurations with vorticity  $k$  (vortex number), the energy of is bounded from below by the following BPS bound

$$E \geq kM_v \equiv -\xi \int d^2x \text{Tr}[f_{12} t^0] = 2\pi ck, \quad k \in \mathbb{Z}, \quad (2.8)$$

where we have assume that the last term in Eq. (2.7) vanishes at infinity. This bound is saturated if the following BPS equations are satisfied:

$$\mathcal{D}_{\bar{z}} H = 0, \quad \frac{2}{e^2} f_{12} t^0 + \frac{2}{g^2} F_{12}^a t^a = H H^\dagger - c\mathbf{1}_N, \quad (2.9)$$

where  $z = x^1 + ix^2$  is a complex coordinate. One can easily verify that all the solutions of the BPS equations solve the original equations of motion of the Lagrangian in Eq. (2.1). The integration constants (the moduli parameters or the collective coordinates) contained in the solutions of the BPS equations parameterize the set of configurations with degenerate energy, that is, the moduli space of BPS vortices  $\mathcal{M}_k$ . There are  $N$  complex moduli parameters for each vortex: one of  $N$  is position  $z_I$  and the rest  $N - 1$  are internal orientations  $\vec{\beta}_I$  ( $I = 1, 2, \dots, k$ ). Since no net forces are exerted among static vortices, each vortex has the position moduli  $z_I$  as its degree of freedom. The internal orientation  $\vec{\beta}_I$  is associated with the  $SU(N)_{C+F}$  color-flavor symmetry, broken by each vortex down to  $SU(N - 1) \times U(1)$ . The Nambu-Goldstone zero modes localize on each vortex and the corresponding moduli  $\vec{\beta}_I$  parameterize the coset

$$\frac{SU(N)}{SU(N - 1) \times U(1)} \cong \mathbb{C}P^{N-1}. \quad (2.10)$$

In the following, the  $(N - 1)$ -dimensional vector  $\vec{\beta}_I$  denotes the inhomogeneous coordinates of  $\mathbb{C}P^{N-1}$  for the internal orientation of  $I$ -th vortex. The moduli space  $\mathcal{M}_k$  is a  $kN$ -dimensional Kähler manifold parameterized by the holomorphic coordinates  $z_I$  and  $\vec{\beta}_I$  and has the  $SU(N)$  isometry acting on  $\vec{\beta}_I$ , which descends from the  $SU(N)_{C+F}$  global symmetry in the vacuum.

## 2.2 Asymptotic metric for non-Abelian vortices

The low-energy dynamics of the vortex system can be described by an effective Lagrangian in which the moduli parameters are promoted to dynamical variables. The effective Lagrangian for these moduli parameters is given in terms of the Kähler metric on the moduli space

$$L = g_{i\bar{j}} \dot{\phi}^i \dot{\bar{\phi}}^{\bar{j}}, \quad (i, j = 1, \dots, \dim_{\mathbb{C}} \mathcal{M}_k = kN), \quad (2.11)$$

where  $\phi^i$  are the holomorphic coordinates on the moduli space:

$$\{\phi^i\} = \{z_I, \vec{\beta}_I\}. \quad (2.12)$$

The metric  $g_{i\bar{j}}$  of the moduli space consists of the free part and the interaction part which are given in terms of the corresponding Kähler potentials

$$g_{i\bar{j}} \equiv \frac{\partial^2}{\partial \phi^i \partial \bar{\phi}^{\bar{j}}} K_{\text{free}} + \frac{\partial^2}{\partial \phi^i \partial \bar{\phi}^{\bar{j}}} K_{\text{int}}. \quad (2.13)$$

The free part describes the dynamics of completely isolated vortices<sup>2</sup>

$$K_{\text{free}} = \sum_{I=1}^k \left[ \frac{1}{2} M_v |z_I|^2 + \frac{4\pi}{g^2} \log(1 + |\vec{\beta}_I|^2) \right], \quad (2.14)$$

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<sup>2</sup> The Kähler class  $4\pi/g^2$  can be determined [42] from the fact that sigma model instantons inside a vortex worldsheet are Yang-Mills instantons from the bulk point of view [43].



$m_g/m_e$	$c_e$	$c_g$
0		1.1363(7)
0.25		1.1853(1)
0.5		1.3090(5)
0.75	2.1955(9)	1.48517(9)
1	1.7078(6)	1.7078(6)
1.5	1.4714(7)	2.3031(0)
2	1.4036(9)	3.14(5)
2.5	1.3746(1)	4.31(8)
3	1.3594(3)	5.9(5)
$\infty$	1.3266(7)	

Table 1: Numerical data for  $k = 1$   $U(2)$  vortex. The constants  $c_e$  and  $c_g$  are not well-defined for  $m_e < 2m_g$  and  $m_g \rightarrow \infty$ , respectively [44].

where  $M_v = 2\pi c$  is the tension of the vortex and  $4\pi/g^2$  corresponds to the radius of  $\mathbb{C}P^{N-1}$ . On the other hand, the interaction part of the Kähler potential describes the leading interactions between well-separated vortices [40]

$$K_{\text{int}} \approx \sum_{I < J} K^{(I,J)}, \quad K^{(I,J)} \equiv -2\pi N \left[ \frac{c_e^2}{e^2} K_0(m_e |z_{IJ}|) + \frac{c_g^2}{g^2} \Theta_{IJ} K_0(m_g |z_{IJ}|) \right], \quad (2.15)$$

where  $K_0$  stands for the modified Bessel function of the second kind. The interaction term  $K^{(I,J)}$  between  $I$ -th and  $J$ -th vortices is a function of the relative distance and an  $SU(N)$  invariant quantity  $\Theta_{IJ}$  defined, respectively, by

$$|z_{IJ}| \equiv |z_I - z_J|, \quad \Theta_{IJ} \equiv N \frac{|1 + \vec{\beta}_I^\dagger \cdot \vec{\beta}_J|^2}{(1 + |\vec{\beta}_I|^2)(1 + |\vec{\beta}_J|^2)} - 1. \quad (2.16)$$

The origin of the modified Bessel function  $K_0$  appearing in the interaction term can be traced back to the asymptotic tail of a profile function of the vortex. Since the leading term in the modified Bessel function  $K_0(m|z_{IJ}|)$  is of order  $e^{-m|z_{IJ}|}$ , the interactions exponentially vanish for large  $|z_{IJ}|$ . Note that the mass scales  $m_e$  and  $m_g$  can be interpreted as inverse widths of “Abelian core” and “non-Abelian core” of a vortex, respectively. The strength of the asymptotic coupling is controlled by the constants  $c_e$  and  $c_g$ , which depend on the ratio  $m_g/m_e$  and  $N$  [44]. The numerical values of  $c_e$  and  $c_g$  for  $U(2)$  vortices are given in Table 1.

For later convenience, let us factor out  $4\pi/g^2$  from the Kähler potential and rewrite it in terms of three coupling constants  $g, m_e$  and  $m_g$  instead of  $e, g, c$

$$K_{\text{free}} = \frac{4\pi}{g^2} \sum_{I=1}^k \left[ \frac{1}{4} m_g^2 |z_I|^2 + \log(1 + |\vec{\beta}_I|^2) \right], \quad (2.17)$$

$$K_{\text{int}} \approx \frac{4\pi}{g^2} \sum_{I < J} \left[ -\frac{N}{2} m_g^2 \left( \frac{c_e^2}{m_e^2} K_0(m_e |z_{IJ}|) + \frac{c_g^2}{m_g^2} \Theta_{IJ} K_0(m_g |z_{IJ}|) \right) \right]. \quad (2.18)$$

### 2.3 Equation of motion along geodesics

The overall coefficient  $1/g^2$  of the effective Lagrangian plays a role of the Planck constant (loop counting parameter) for fixed mass scales  $m_g$  and  $m_e$ . Therefore, the quantum effects are negligible and the classical analysis of the effective Lagrangian is valid for the energy scale  $E$  much larger than  $g^2$

$$g^2 \ll E \ll m_g, m_e. \quad (2.19)$$

Note that the energy scale  $E$  should be much smaller than mass gaps to justify the use of the low-energy effective Lagrangian. In the classical analysis, the dynamics are completely independent of the gauge coupling  $g$ .

The equations of motion for the moduli parameters  $\phi^i$  take the following form of the geodesic equation

$$\ddot{\phi}^i + \Gamma_{jk}^i \dot{\phi}^j \dot{\phi}^k = 0, \quad \Gamma_{jk}^i \equiv g^{\bar{l}i} \partial_j g_{k\bar{l}}. \quad (2.20)$$

Since the interactions are sufficiently small ( $K^{(I,J)} \approx e^{-m|z_{IJ}|}$ ) for well-separated vortices, the Christoffel symbol  $\Gamma_{jk}^i$  can be approximated as

$$\Gamma_{jk}^i \approx \hat{\Gamma}_{jk}^i + \hat{g}^{\bar{l}i} \hat{\nabla}_j \partial_k \partial_{\bar{l}} K_{int}, \quad (2.21)$$

where  $\hat{\Gamma}_{jk}^i$ ,  $\hat{g}_{i\bar{j}}$  and  $\hat{\nabla}_i$  are the free part of the Christoffel symbol, metric and covariant derivative respectively. Then, the equations of motion for  $z_I$  can be written as

$$\ddot{z}_I = \frac{2}{m_g^2} \sum_{J \neq I} \frac{\partial}{\partial \bar{z}_{IJ}} \delta_{IJ} K^{(I,J)}, \quad (2.22)$$

where we have defined a differential operator  $\delta_{IJ}$  by

$$\delta_{IJ} \equiv \dot{z}_{IJ}^2 \frac{\partial^2}{\partial z_{IJ}^2} + 2\dot{z}_{IJ} \frac{\partial}{\partial z_{IJ}} \left( \dot{\vec{\beta}}_I \cdot \frac{\partial}{\partial \vec{\beta}_I} + \dot{\vec{\beta}}_J \cdot \frac{\partial}{\partial \vec{\beta}_J} \right) + 2\dot{\vec{\beta}}_I \cdot \frac{\partial}{\partial \vec{\beta}_I} \dot{\vec{\beta}}_J \cdot \frac{\partial}{\partial \vec{\beta}_J}. \quad (2.23)$$

The equations of motion for the orientations are

$$\frac{\hat{\nabla}_t \dot{\vec{\beta}}_I}{1 + |\vec{\beta}_I|^2} = \sum_{J \neq I} \left[ \left( \frac{\partial}{\partial \vec{\beta}_I^\dagger} + \vec{\beta}_I \vec{\beta}_I^\dagger \cdot \frac{\partial}{\partial \vec{\beta}_I^\dagger} \right) \delta_{IJ} - \frac{2\dot{\vec{\beta}}_I}{1 + |\vec{\beta}_I|^2} \dot{\vec{\beta}}_I \cdot \frac{\partial}{\partial \vec{\beta}_I} \right] K^{(I,J)}, \quad (2.24)$$

where  $\hat{\nabla}_t \dot{\vec{\beta}}_I$  is the free part of the covariant derivative along the trajectory on  $\mathbb{CP}^{N-1}$

$$\hat{\nabla}_t \dot{\vec{\beta}}_I \equiv \ddot{\vec{\beta}}_I - \frac{2}{1 + |\vec{\beta}_I|^2} (\vec{\beta}_I^\dagger \cdot \dot{\vec{\beta}}_I) \dot{\vec{\beta}}_I. \quad (2.25)$$

We refer the right-hand sides of the equations of motion (2.22) and (2.24) as forces, more precisely, geodesic forces. In general, they are proportional to squares of velocities  $\propto \partial_t \phi^i \partial_t \phi^j$ .

### 3 Scattering of $U(2)$ non-Abelian vortices

In this section, we discuss the asymptotic scattering of the non-Abelian vortices in the simplest example of the  $U(2)$  case which shows essential differences between the Abelian and the non-Abelian vortices. In this case, the orientational moduli space becomes a sphere  $S^2 \simeq \mathbb{C}P^1$ . The inhomogeneous coordinate  $\beta_I$  is given by the stereographic projection from the north pole of  $S^2$  and related to the standard spherical coordinates as

$$\beta_I = \tan \frac{\theta_I}{2} e^{i\varphi_I}. \quad (3.1)$$

One can also make use of the following three-dimensional unit vector as coordinates of  $S^2$

$$\vec{n}_I = (\sin \theta_I \cos \varphi_I, \sin \theta_I \sin \varphi_I, \cos \theta_I). \quad (3.2)$$

Since  $SO(3) \simeq SU(2)/\mathbb{Z}_2$  symmetry is manifest on  $\vec{n}_I$ , the dynamics of the orientational modes can be better understood in terms of  $\vec{n}_I$ . The Lagrangian written in terms of  $\vec{n}_I$  is given in Appendix A.

#### 3.1 Noether charge

One of the sharp contrast of the dynamics of non-Abelian vortex to that of Abelian vortex comes from the internal orientations. As mentioned in the introduction, such internal degrees of freedom leads to conserved Noether charges. So let us begin with describing the conserved charges of non-Abelian vortices.

Corresponding to the  $SU(2)$  global symmetry, we have one set of conserved charges  $\vec{Q}$ . In the original theory, they are given by

$$\vec{Q} \equiv \frac{i}{2} \int d^2x \operatorname{Tr} \left[ H \vec{\sigma} \mathcal{D}_0 H^\dagger - \mathcal{D}_0 H \vec{\sigma} H^\dagger \right], \quad (3.3)$$

where  $\vec{\sigma}$  are the Pauli matrices. Although  $\vec{Q} = 0$  for static non-Abelian vortices, these charges arise from the motion of the orientational modes.

In the effective action of vortices, the charges of vortices are given by

$$\vec{Q} = \frac{\partial^2 K}{\partial \beta^I \partial \bar{\phi}^j} \vec{\xi}^I \dot{\bar{\phi}}^j + (\text{c.c.}), \quad (3.4)$$

where  $\vec{\xi}^I$  are the holomorphic Killing vectors associated with  $SU(2)$  rotation on the moduli space. Explicit form of the  $SU(2)$  triplet vector is given by

$$\vec{\xi} = \sum_{I=1}^k \vec{\xi}^I \frac{\partial}{\partial \beta_I}, \quad \vec{\xi}^I \equiv \left( -\frac{i}{2}(1 - \beta_I^2), \frac{1}{2}(1 + \beta_I^2), i\beta_I \right). \quad (3.5)$$

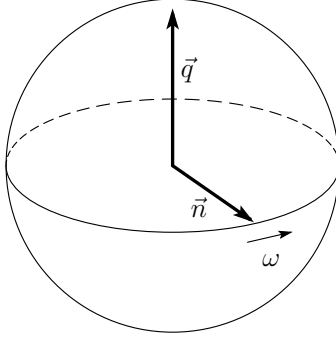


Fig. 1: The free motion of the internal orientation on  $\mathbb{CP}^1 \simeq S^2$ .

These charges have contributions from the free and interaction parts of the effective Lagrangian. If all the vortices are isolated, contributions from the free part of the Lagrangian for each individual vortex are separately conserved. In terms of  $\vec{n}_I$ , we define the charge of the  $I$ -th vortex by<sup>3</sup>

$$\vec{Q}_I \equiv \frac{2\pi}{g^2} \vec{n}_I \times \dot{\vec{n}}_I. \quad (3.6)$$

For the isolated vortices, the dynamics of the orientation is described by free equation of motion, which is nothing but the conservation law of the charge of each individual vortex

$$\frac{d}{dt} \vec{Q}_I = \frac{2\pi}{g^2} \vec{n}_I \times \ddot{\vec{n}}_I = 0. \quad (3.7)$$

This equation is equivalent to the geodesic equation on a sphere, so that the trajectory of the orientation is a great circle, as shown in Fig. 1:

$$\vec{n}_I = \vec{n}_{I0} \cos(\omega_I t) + \vec{q}_I \times \vec{n}_{I0} \sin(\omega_I t), \quad (3.8)$$

where  $\vec{n}_{I0}$  and  $\vec{q}_I$  are unit constant vectors satisfying  $\vec{n}_{I0} \cdot \vec{q}_I = 0$ . The charge of this solution is given by

$$\vec{Q}_I = \frac{2\pi\omega_I}{g^2} \vec{q}_I. \quad (3.9)$$

Note that we can always set  $\omega_I \geq 0$  without loss of generality.

### 3.2 Features of the scattering of non-Abelian vortices

As stressed in the previous subsection, the internal orientations and the conserved Noether charges make the dynamics of non-Abelian vortices quite different from that of the well-known

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<sup>3</sup> The normalization of  $\vec{Q}_I$  is chosen so that they obey the half-integer quantization condition in the quantum theory. Although the coupling constant  $g$  appears in the classical equations of motion because of this normalization, it can be absorbed by rescaling  $\vec{Q}_I$  appropriately.

ANO vortices. Since the internal orientations are continuous parameters, there exist continuously different initial conditions, which can make the dynamics quite complicated. In order to see how much the dynamics of non-Abelian vortices differs from that of the ANO vortices, we show several numerical solutions for the geodesic equations (2.22) and (2.24).

The relative internal orientations between the first and second vortices are denoted as  $\Delta\theta = \theta_1 - \theta_2$ ,  $\Delta\varphi = \varphi_1 - \varphi_2$ . We specify initial conditions at sufficiently past. We can always use the  $SU(2)$  symmetry to set the initial values at sufficiently past as

$$\varphi_{10} = -\varphi_{20}, \quad \dot{\varphi}_{10} = \dot{\varphi}_{20} = 0. \quad (3.10)$$

This initial condition corresponds to a pair of vortices whose orientations are rotating around great circles at the longitudes  $\varphi_{10}$  and  $\varphi_{20} = -\varphi_{10}$ . Therefore, we can choose five initial conditions for the orientations:  $\theta_I, \dot{\theta}_I$  ( $I = 1, 2$ ) and  $\Delta\varphi$ .

### Scattering of vortices without initial $Q$ -charges

As the first example of the scattering of two non-Abelian vortices, we examine the case of  $m_e = m_g$  and choose the following initial conditions for the relative orientation  $\Delta\varphi_0$  and for the velocities of the orientations  $\dot{\theta}_{I0}$ :

$$\Delta\varphi_0 = 0, \quad \dot{\theta}_{I0} = 0, \quad (I = 1, 2). \quad (3.11)$$

In this setting, the vortices have no initial charges  $\vec{Q}_{I0} = 0$ . Because of these initial conditions, the variation of the internal orientation is very small before the vortices passes through the interaction region.

In Fig. 2-(a), we show the scattering orbits for the initial relative orientation  $\Delta\theta_0 = \pi/2$  by changing the impact parameter  $a$  as  $a = 2.0, 2.5, 3.0, \dots$ . One can clearly see that the moving vortices feel the repulsive geodesic force between each other. At a glance, this result appears to be very similar to the scattering of the ANO vortices, in which case the scattering is repulsive and the orbits are uniquely determined if the initial velocity and the impact parameter are fixed. However, it is actually different. See Fig. 2-(b) where the impact parameter is fixed at  $a = 3$  and the initial relative orientation is varied as  $\Delta\theta_0 = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$ . When  $\Delta\theta_0 = 0$  ( $\Delta\theta_0 = \frac{\pi}{4}, \frac{\pi}{2}, \dots$ ), they scatter (almost) in the same way as the Abelian vortices. On the other hand, non-Abelian vortices just pass through each other without feeling any interactions for  $\Delta\theta_0 = \pi$  (a pair of vortices with anti-parallel orientations). This behavior is markedly different from the scattering of ANO vortices. We thus have found that the scattering angle is sensitive to the relative orientation.

The dependence on the relative orientation can be roughly understood from the fact that the interaction part of the Kähler potential  $K_{\text{int}}$  given in Eq. (2.18) is proportional to  $1 + \Theta_{12}$

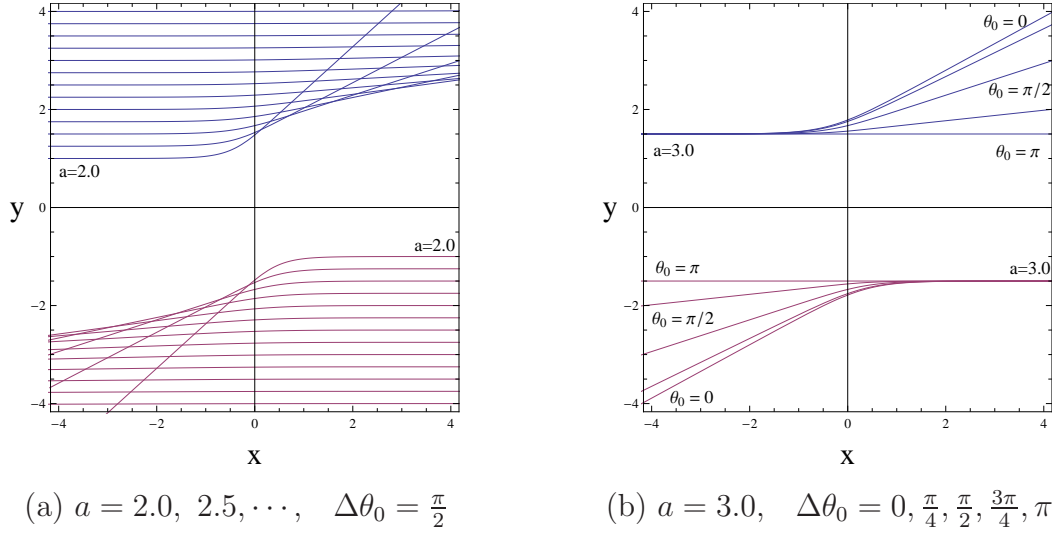


Fig. 2: Scattering orbits of non-Abelian vortices on the  $z$ -plane ( $(x, y)$ -plane) in the case of  $m_e = m_g = 1$  ( $c_e = c_g = 1.708$ ), for (a) various values of the impact parameter  $a$  with the fixed initial relative orientation  $\Delta\theta_0 = \frac{\pi}{2}$ , and (b) various values of the initial relative orientation  $\Delta\theta_0$  with the fixed impact parameter  $a = 3.0$ . In the both cases the initial velocities of the orientations are zero ( $\vec{Q}_I = 0$ ). No interaction exists for  $\Delta\theta_0 = \pi$  (anti-parallel orientations). Otherwise, the interaction is always repulsive with reaching the maximum at  $\Delta\theta_0 = 0$  (parallel orientations), where the interaction reduces to that of the ANO vortices.

for  $m_g = m_e$  ( $c_g = c_e$ ). Since the relative orientation is almost unchanged from the initial condition until the vortices leave the interaction region, it follows that the scattering angle is also proportional to  $1 + \Theta_{12}$ . For example, the scattering angle is maximized (vanishes) for  $\Delta\theta_0 = 0$ , ( $\Delta\theta = \pi$ ) at which  $1 + \Theta_{12} = 2$ , ( $1 + \Theta_{12} = 0$ ). We emphasize again that, in contrast to the non-Abelian vortex scattering, the scattering of the ANO vortices is uniquely determined with the initial velocity and the impact parameter fixed.

In this example, we have taken the initial condition without initial  $Q$ -charges. The numerical calculations show that the vortices are charged  $\vec{Q}_1 = -\vec{Q}_2 \neq 0$  after the scattering, even if orientational moduli is initially static. We will see this phenomenon via an analytic discussion in section 3.4.

### Scattering of vortices with maximized non-Abelian effect

As the second example let us consider the case where non-Abelian effects are maximized. If the Abelian vector boson mass is sent to infinity<sup>4</sup>  $m_e \rightarrow \infty$ , the Abelian part of the interaction

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<sup>4</sup> Our original model reduces in the limit  $m_e \rightarrow \infty$  to a  $\mathbb{CP}^1$  nonlinear sigma model whose  $SU(2)$  isometry is gauged.

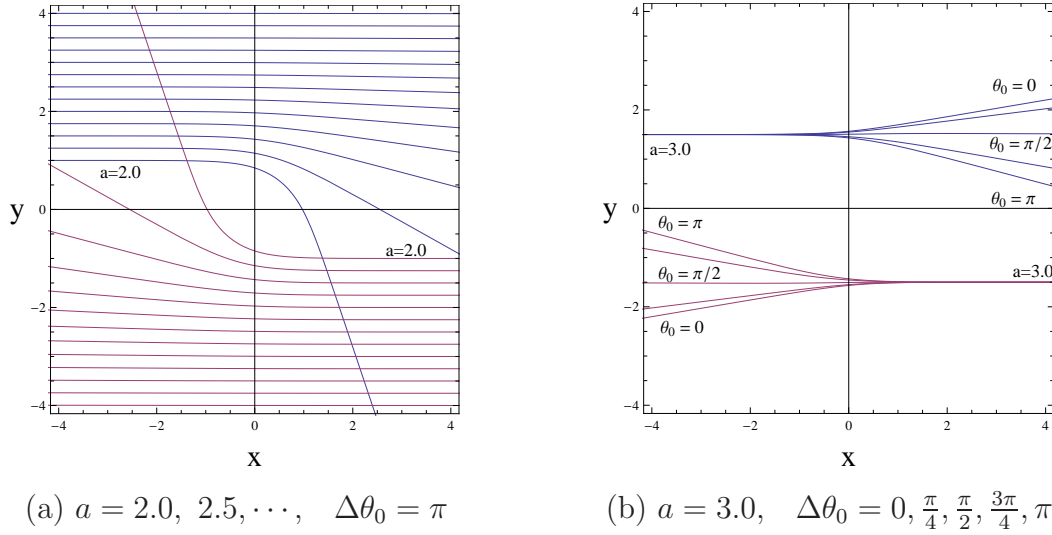


Fig. 3: Scattering orbits of non-Abelian vortices in the  $z$ -plane ( $(x, y)$ -plane) for  $m_e = \infty, m_g = 1$  ( $c_g = 1.136$ ), with (a) various values of the impact parameter  $a$  and the fixed initial relative orientation  $\Delta\theta_0 = \pi$ , and (b) various values of the initial relative orientation  $\Delta\theta_0$  and the fixed impact parameter  $a = 3.0$ . In the both cases the other initial conditions are:  $\Delta\varphi = 0$  and  $\dot{\theta}_I = 0$ .

Kähler potential  $K_{\text{int}}$ , given by the first term in Eq.(2.18), is highly suppressed and only the non-Abelian part, the second term, survives. The numerical results are shown in Fig. 3. Except for the Abelian mass  $m_e$ , the other parameters and the initial conditions are chosen to be the same as those for Fig. 2. Fig. 3-(a) shows the scattering orbit for  $\Delta\theta_0 = \pi$  where the vortices clearly *attract* each other. This attractive force is a characteristic property of the non-Abelian case, which has not been seen in the Abelian case. As before, the scattering angle is affected by the initial relative angle of the internal orientations, as shown in Fig. 3-(b). In the present case, the interaction is proportional to  $\Theta_{12} = \{1, 0, -1\}$  for  $\Delta\theta_0 = \{0, \frac{\pi}{2}, \pi\}$ , respectively. Thus, the scattering angle changes its sign at  $\Delta\theta_0 = \frac{\pi}{2}$ . For  $\Delta\theta_0 < \frac{\pi}{2}$ , the vortices repel as in the Abelian case, while they attract for  $\Delta\theta_0 > \frac{\pi}{2}$ . The parallel orientation ( $\Delta\theta_0 = 0$ ) gives the maximal repulsion and the anti-parallel orientation ( $\Delta\theta_0 = \pi$ ) gives the maximal attraction.

### Scattering of vortices with non-zero $Q$ -charges

The last interesting example is the scattering of non-Abelian vortices which have non-zero initial charges  $\vec{Q}_I$ . Since a repulsive (attractive) force works between the same (opposite) charges, the scattering is very different from that with vanishing charges. The results are shown in Fig. 4, where  $\gamma_I \equiv \dot{\theta}_{I0}/(m_g v)$  is defined by the initial relative velocity  $v$  and the angular velocities  $\dot{\theta}_{I0}$ . Fig. 4-(a) shows the scattering orbits of vortices with the same initial  $Q$ -charges. Because of the repulsive force, the vortices recoil even for  $a = 0$ . On the other hand, the vortices with opposite

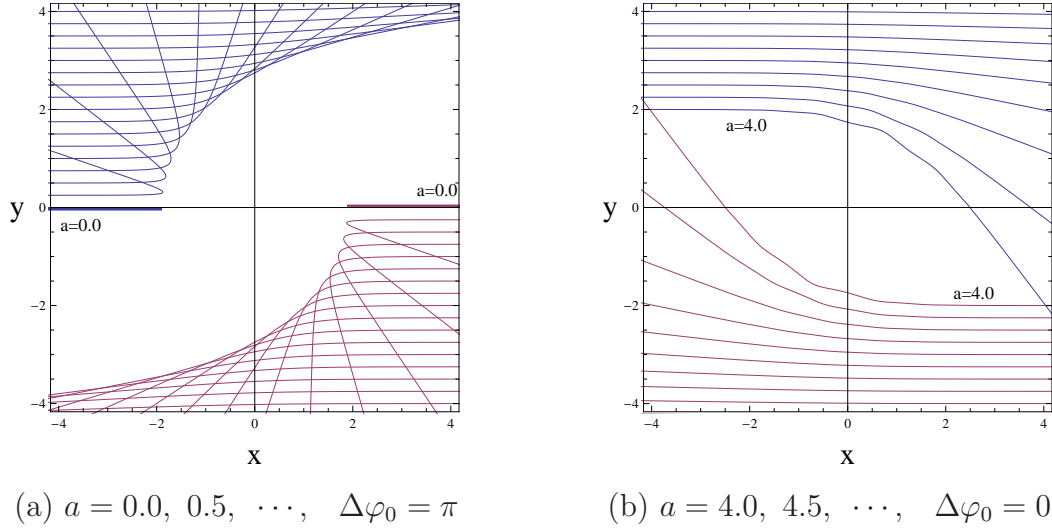


Fig. 4: Scattering orbits of non-Abelian vortices with non-vanishing conserved charges for  $\gamma_1 = -\gamma_2 = \frac{10}{3}$ ,  $m_e = m_g = 1$  ( $c_e = c_g = 1.708$ ), with various values of the impact parameter  $a$  and of the initial relative orientation  $\Delta\varphi_0$ . The other initial conditions are :  $\theta_{10} = \theta_{20} = \frac{\pi}{2}$ .

charges feel attractive force as shown in Fig. 4-(b). We can see that their orbits are slightly wavy. This is due to oscillations of the forces caused by the rotations of the orientations. In the next section, we will discuss an “average” over the rapid motions of the orientations to find out effective forces between non-Abelian vortices with  $Q$ -charges.

### 3.3 Forces between non-Abelian vortices

Now let us examine the geodesic forces between two vortices induced by motions of the moduli parameters. To this end, let us first rewrite the equations of motion Eqs. (2.22) and (2.24) in terms of  $\vec{n}_I$  as<sup>5</sup>

$$\begin{aligned} \ddot{z}_{12} = & - \left[ \frac{c_e^2}{m_e} K_1(m_e |z_{12}|) + \frac{c_g^2}{m_g} (\vec{n}_1 \cdot \vec{n}_2) K_1(m_g |z_{12}|) \right] \frac{\bar{z}_{12}}{|z_{12}|} (m_g \dot{z}_{12})^2 \\ & + \frac{2c_g^2}{m_g} K_0(m_g |z_{12}|) (\vec{n}_1 \cdot \vec{\alpha}_2 + \vec{n}_2 \cdot \vec{\alpha}_1) m_g \dot{z}_{12} - \frac{2c_g^2}{m_g} K_1(m_g |z_{12}|) \vec{\alpha}_1 \cdot \vec{\alpha}_2 \frac{z_{12}}{|z_{12}|}, \end{aligned} \quad (3.12)$$

<sup>5</sup> Note that the equation of motion  $\vec{n} \times \ddot{\vec{n}} = \vec{A}$  is equivalent to  $\ddot{\vec{n}} = \vec{A} \times \vec{n} - \vec{n} |\dot{\vec{n}}|^2$  for vectors  $\vec{n}$  and  $\vec{A}$  such that  $|\vec{n}|^2 = 1$ ,  $\vec{n} \cdot \vec{A} = 0$ .



$$\begin{aligned}
\vec{n}_1 \times \ddot{\vec{n}}_1 = & \operatorname{Re} \left[ \frac{1}{2} c_g^2 K_2(m_g |z_{12}|) \vec{n}_1 \times (\vec{n}_2 - i \vec{n}_1 \times \vec{n}_2) \left( \frac{\bar{z}_{12}}{|z_{12}|} m_g \dot{z}_{12} \right)^2 \right. \\
& + 2 c_g^2 K_1(m_g |z_{12}|) i \vec{\alpha}_1 \left( \vec{n}_1 \cdot \vec{n}_2 - \frac{\vec{\alpha}_1^\dagger \cdot \vec{\alpha}_2}{|\vec{\alpha}_1|^2} \right) \frac{\bar{z}_{12}}{|z_{12}|} m_g \dot{z}_{12} \\
& \left. - c_g^2 K_0(m_g |z_{12}|) i \vec{\alpha}_1 (\vec{n}_2 \cdot \vec{\alpha}_1 + 2 \vec{n}_1 \cdot \vec{\alpha}_2) \right]. \tag{3.13}
\end{aligned}$$

where  $\vec{\alpha}_I$  ( $I = 1, 2$ ) is a complex three-vector defined by

$$\vec{\alpha}_I \equiv \dot{\vec{n}}_I - i \vec{n}_I \times \dot{\vec{n}}_I. \tag{3.14}$$

Note that the equation of motion for  $\vec{n}_2$  can be obtained by exchanging  $\vec{n}_1$  and  $\vec{n}_2$  in Eq. (3.13).

The geodesic force given in the right-hand side of Eq. (3.12) shows that the motion in orientational moduli space generally induces a force between the vortices even if there is no spatial motion initially. Similarly, Eq. (3.13) shows that a spatial motion induces a motion in orientational moduli space, even if orientational moduli is initially static. This implies that the kinetic energy in spatial motion and in orientational moduli transmute each other.

### 3.3.1 Average over rapid motion of internal orientations

As we have seen in the numerical example Fig. 4-(b), the sign and magnitude of forces between them oscillate due to the rotations of  $\vec{n}_I$ . Since motion in orientational moduli is (almost) periodic, its physical effect is best seen by averaging over the periods of two orientational moduli individually. Denoting this time average for the high-frequency modes by  $\langle \rangle$ , we can express the averaged equation of motion for the relative position  $z_{12}$  as

$$\langle \ddot{z}_{12} \rangle = \left\langle -\frac{z_{12}}{|z_{12}|} \left( c_e^2 m_e K_1(m_e |z_{12}|) \left( \frac{\dot{z}_{12} \bar{z}_{12}}{|z_{12}|} \right)^2 - \frac{c_g^2 g^4}{2\pi^2 m_g} K_1(m_g |z_{12}|) \vec{Q}_1 \cdot \vec{Q}_2 \right) \right\rangle. \tag{3.15}$$

This approximation is valid only when the motion in space is slow compared to the velocities of the orientations,

$$\omega_1, \omega_2 \gg m_g |\dot{z}_{12}| \quad \text{and} \quad \omega_1 \not\approx \omega_2. \tag{3.16}$$

. The first term depends only on the velocities of the vortices just in the same way as the force between Abelian vortices. The second term is the dominant force induced by the motion of orientational moduli. Note that the order of the first term is  $m_e |\dot{z}_{12}|^2$ , while that of the second term is  $\omega_1 \omega_2 / m_g$ . Thus, in the case of  $m_e \geq m_g$ , the first term is negligible compared to the second term under the condition (3.16). On the other hand, if  $m_e < m_g$ , the first term can be dominant since  $K_0(m_e |z_{12}|) \gg K_0(m_g |z_{12}|)$  asymptotically.

It has been shown that the interaction of non-Abelian vortices are well described by regarding them as point-like sources of the Higgs fields and massive vector fields [40]. The motion of the orientation of the  $I$ -th vortex induces the following (color) electric charge distribution<sup>6</sup> and the massive vector field

$$j_0^I = \frac{2g^2c_g}{m_g^2} \vec{Q}_I \cdot \vec{\sigma} \partial_z \partial_{\bar{z}} \delta^2(z - z_I), \quad W_0^I = \frac{g^2c_g}{4\pi} \vec{Q}_I \cdot \vec{\sigma} K_0(m_g|z - z_I|). \quad (3.17)$$

Therefore, the electrostatic potential between  $I$ -th and  $J$ -th vortices is given by

$$V_{IJ} = \int d^2x \text{Tr} \left[ \frac{1}{g^2} j_0^I W_0^J \right] = \frac{c_g^2 g^2}{4\pi} \vec{Q}_I \cdot \vec{Q}_J K_0(m_g|z_{IJ}|). \quad (3.18)$$

The second term in the averaged equation of motion Eq. (3.15) can be attributed to this “electrostatic interaction”. This force is repulsive (attractive) if the inner product of the charges of two vortices is positive (negative). Namely the vortices with aligned charges repel each other, whereas those with disaligned charges attract each other. These phenomena can be seen in the numerical calculations in Fig. 4.

Let us next investigate the averaged equations of motion for the orientations. By using the charge vectors  $\vec{Q}_I$  defined in Eq. (3.6), the equation of motion of  $\vec{n}_1$  Eq. (3.13) reduces to

$$\left\langle \frac{d\vec{Q}_1}{dt} \right\rangle = \frac{g^2 c_g^2}{2\pi} \left\langle K_0(m_g|z_{12}|) (\vec{Q}_2 \times \vec{Q}_1) \right\rangle, \quad (3.19)$$

where we have used  $\langle \vec{n}_I \otimes \vec{n}_I^T \rangle = \frac{1}{2} \langle \mathbf{1} - \vec{q}_I \otimes \vec{q}_I^T \rangle$ . The equation for  $\vec{Q}_2$  can be obtained just by exchanging the two vortices ( $1 \leftrightarrow 2$ ). These equations describe precessions of the orientations, that is, motions along great circles with slowly moving axes. We can see from Eq. (3.19) that the sum  $\vec{Q}_{\text{tot}} \equiv \langle \vec{Q}_1 + \vec{Q}_2 \rangle$  and the inner product  $\langle \vec{Q}_1 \cdot \vec{Q}_2 \rangle$  are conserved. By solving averaged equation of motion, we find that each charge  $\vec{Q}_I$  slowly rotates around  $\vec{Q}_{\text{tot}}$  as (see Fig. 5)

$$\langle \vec{Q}_1 \rangle \simeq a \vec{Q}_{\text{tot}} + \vec{Q}_{\text{osc}} \cos \theta(t) + \frac{\vec{Q}_{\text{tot}} \times \vec{Q}_{\text{osc}}}{|\vec{Q}_{\text{tot}}|} \sin \theta(t), \quad (3.20)$$

$$\langle \vec{Q}_2 \rangle \simeq b \vec{Q}_{\text{tot}} - \vec{Q}_{\text{osc}} \cos \theta(t) - \frac{\vec{Q}_{\text{tot}} \times \vec{Q}_{\text{osc}}}{|\vec{Q}_{\text{tot}}|} \sin \theta(t), \quad (3.21)$$

where  $a + b = 1$ ,  $\vec{Q}_{\text{osc}} \cdot \vec{Q}_{\text{tot}} = 0$  and

$$\theta(t) = \frac{g^2 c_g^2}{2\pi} |\vec{Q}_{\text{tot}}| \int^t dt K_0(m_g|z_{12}|). \quad (3.22)$$

We can see that the angular frequency of the rotation is quite small  $\dot{\theta} \ll \omega_{1,2}$ .

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<sup>6</sup> The charges  $\vec{Q}_I$  themselves are not the electric charges but the charges of  $SU(2)$  global symmetry. The total electric charges are always zero even for vortices with rotating orientations.

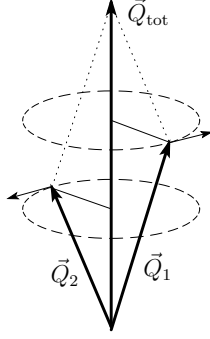


Fig. 5: An example of motions of the charge vectors  $\vec{Q}_I$  for  $\omega_I \gg m_g |\dot{z}_{12}|$ .

In contrast to the Abelian vortex, the non-Abelian vortex has the characteristic feature that the interactions depend on the relative internal orientations. Since the inner product  $\vec{Q}_1 \cdot \vec{Q}_2$  is almost conserved during the scattering process, the interaction can be, for instance, kept to be attractive by choosing near anti-parallel orientations. In such a situation, one can imagine a possibility for two non-Abelian vortices to be bound together. However, it is not the case. It is because the balanced point for the exponentially decaying attractive force ( $\sim e^{m_g |z_{12}|}$ ) with the centrifugal force ( $\sim 1/|z_{12}|^2$ ) is unstable. Of course, this conclusion is valid only for the well-separated vortices at distances  $|z_{12}| > 1/m_g$ . Although one may expect that bound states can exist when  $|z_{12}| < 1/m_g$ , such a region is out of range of validity of our approximation. In Sec. 4, we will see that a bound state of two (dyonic) vortices exists in a mass deformed theory where a certain extra force, which is not a geodesic force, is induced by a mass term.

### 3.3.2 Limit of slowly moving internal orientations

Let us next consider the opposite limit of Eq. (3.16),

$$\omega_I \ll m_g |\dot{z}_{12}|. \quad (3.23)$$

Now the motion of the internal orientations is very slow compared to the spatial motion, so that we can safely neglect  $\dot{\vec{n}}_I$  in the equations. Thus the first lines of Eq. (3.12) and Eq. (3.13) give dominant contributions. Depending on the mass scales  $m_g$  and  $m_e$ , the dominant contributions in Eq. (3.12) are given by

$$\ddot{z}_{12} \approx \begin{cases} -\frac{c_e^2}{m_e} K_1(m_e |z_{12}|) \frac{\bar{z}_{12}}{|z_{12}|} (m_e \dot{z}_{12})^2 & \text{for } m_e < m_g \\ -\frac{c_e^2}{m_e} (1 + \vec{n}_1 \cdot \vec{n}_2) K_1(m_e |z_{12}|) \frac{\bar{z}_{12}}{|z_{12}|} (m_e \dot{z}_{12})^2 & \text{for } m_e = m_g \\ -\frac{c_g^2}{m_g} (\vec{n}_1 \cdot \vec{n}_2) K_1(m_g |z_{12}|) \frac{\bar{z}_{12}}{|z_{12}|} (m_g \dot{z}_{12})^2 & \text{for } m_e > m_g \end{cases} \quad (3.24)$$

Since the interaction for  $m_e < m_g$  is equivalent to that of the Abelian case, the vortices receive repulsive force in the scattering process. For  $m_e > m_g$ , the forces depend on  $\vec{n}_1 \cdot \vec{n}_2$ . When  $\vec{n}_1 \cdot \vec{n}_2 > 0$  ( $\vec{n}_1 \cdot \vec{n}_2 < 0$ ), they repel (attract) each other and the interaction accidentally vanishes at  $\vec{n}_1 \cdot \vec{n}_2 = 0$ . Although the interaction depends on  $\vec{n}_1 \cdot \vec{n}_2$  also in the case of  $m_e = m_g$ , it is qualitatively similar to the Abelian case, that is, the vortices receive the repulsive force except for the case  $\vec{n}_1 \cdot \vec{n}_2 = -1$ . This explains the behaviors of the scattering given in Figs. 2 and 3.

### 3.3.3 Internal orientations at a head-on collision

As a special case, let us consider a pair of vortices which are going to collide head-on with each other

$$\dot{z}_{12} = -v \frac{z_{12}}{|z_{12}|}, \quad \dot{\vec{n}}_1, \dot{\vec{n}}_2 \simeq 0, \quad v > 0. \quad (3.25)$$

From the equation of motion for the orientation Eq. (3.13), we find that

$$\vec{n}_1 \times \ddot{\vec{n}}_1 \simeq c_g^2 m_g^2 v^2 K_2(m_g |z_{12}|) \vec{n}_1 \times \vec{n}_2. \quad (3.26)$$

This equation implies that the orientations of the vortices tend to align before a head-on collision. This result for well-separated vortices naturally extends a similar result obtained by an analysis around a vicinity of coincident vortices [39].

## 3.4 Large impact parameter

If two vortices are completely separated, interactions between them can be neglected, and the solution of the equations of motion is given by the free motion (moving parallel to the  $x$ -axes)

$$z_{12} = vt + ia, \quad \vec{n}_I = \vec{c}_I e^{i\omega_I t} + \vec{c}_I^* e^{-i\omega_I t}. \quad (3.27)$$

Here we have rewritten the solution Eq. (3.8) in terms of a complex vector  $\vec{c}_I$  which is related to  $\vec{n}_{I0}$  and  $\vec{q}_I$  as

$$\vec{c}_I = \frac{1}{2} (\vec{n}_{I0} - i\vec{q}_I \times \vec{n}_{I0}) \quad \leftrightarrow \quad \vec{n}_{I0} = \vec{c}_I + \vec{c}_I^*, \quad \vec{q}_I = -2i\vec{c}_I \times \vec{c}_I^*. \quad (3.28)$$

Let us here consider the case of large impact parameter  $a$ . If two vortices are far apart, their interactions are exponentially suppressed and the deviation from the free motion is small during the scattering process. Therefore we can safely evaluate their interactions by approximating the right-hand side of the equations of motion in Eqs. (3.12) and (3.13) by inserting the free motion in Eq. (3.27).

### 3.4.1 Scattering angle

Substituting the free motion Eq. (3.27) into the right-hand side of Eq. (3.12), we obtain the approximated equation of motion for the relative position  $z_{12}$  in which all the forces are known functions of time. Then, we can evaluate the total change in the relative velocity by integrating the forces from  $t = -\infty$  to  $t = \infty$

$$\Delta \dot{z}_{12} = \int_{-\infty}^{\infty} \ddot{z}_{12} dt. \quad (3.29)$$

This can be carried out explicitly by using the following formulas of Fourier transformations

$$\int_{-\infty}^{\infty} dt K_0(m\sqrt{(vt)^2 + a^2}) e^{i\omega t} = \frac{\pi}{\sqrt{(mv)^2 + \omega^2}} e^{-\frac{a}{v}\sqrt{(mv)^2 + \omega^2}}, \quad (3.30)$$

$$\int_{-\infty}^{\infty} dt \frac{vt \pm ia}{\sqrt{(vt)^2 + a^2}} K_1(m\sqrt{(vt)^2 + a^2}) e^{i\omega t} = \frac{\pi i}{mv} \left[ \frac{\omega}{\sqrt{(mv)^2 + \omega^2}} \pm 1 \right] e^{-\frac{a}{v}\sqrt{(mv)^2 + \omega^2}}. \quad (3.31)$$

Eventually, we obtain the following deviation of the relative velocity

$$\begin{aligned} \frac{\Delta \dot{z}_{12}}{2\pi i v} &= \frac{1}{2} c_e^2 e^{-m_e a} + c_g^2 \gamma_1 \gamma_2 (\vec{q}_1 \cdot \vec{q}_2) e^{-m_g a} \\ &\quad - c_g^2 \gamma_2 \left( \sqrt{1 + \gamma_1^2} \vec{q}_2 + i \gamma_1 \vec{q}_1 \times \vec{q}_2 \right) \cdot \vec{n}_{10} e^{-m_g a \sqrt{1 + \gamma_1^2}} \\ &\quad - c_g^2 \gamma_1 \left( \sqrt{1 + \gamma_2^2} \vec{q}_1 - i \gamma_2 \vec{q}_1 \times \vec{q}_2 \right) \cdot \vec{n}_{20} e^{-m_g a \sqrt{1 + \gamma_2^2}} \\ &\quad + c_g^2 (1 + 2\gamma_1 \gamma_2) \left( \text{Re}(\vec{c}_1 \cdot \vec{c}_2) + \frac{\gamma_1 + \gamma_2}{\sqrt{1 + (\gamma_1 + \gamma_2)^2}} i \text{Im}(\vec{c}_1 \cdot \vec{c}_2) \right) e^{-m_g a \sqrt{1 + (\gamma_1 + \gamma_2)^2}} \\ &\quad + c_g^2 (1 - 2\gamma_1 \gamma_2) \left( \text{Re}(\vec{c}_1 \cdot \vec{c}_2^*) + \frac{\gamma_1 - \gamma_2}{\sqrt{1 + (\gamma_1 - \gamma_2)^2}} i \text{Im}(\vec{c}_1 \cdot \vec{c}_2^*) \right) e^{-m_g a \sqrt{1 + (\gamma_1 - \gamma_2)^2}}. \end{aligned} \quad (3.32)$$

Note that the frequencies  $\omega_I$  appear only through the following ratio

$$\gamma_I \equiv \frac{\omega_I}{m_g v}. \quad (3.33)$$

This follows from the invariance of the ratio  $\Delta \dot{z}_{12}/v$  under the time rescaling  $t \rightarrow \lambda t$ , which arises from the property of the geodesic equation (2.20) itself. This ratio  $\gamma_I$  measures how much the vectors  $\vec{n}_I$  rotate in the typical time scale of the interaction  $\Delta t \approx 1/m_g v$ . For generic values of the ratios  $\gamma_I$  and the vectors  $\vec{q}_I$ , only the first line in Eq. (3.32) is dominant,

$$\Delta \dot{z}_{12} \approx \begin{cases} \pi i v c_e^2 e^{-m_e a} & m_e < m_g \\ 2\pi i v c_g^2 \gamma_1 \gamma_2 (\vec{q}_1 \cdot \vec{q}_2) e^{-m_g a} & m_g < m_e \end{cases}. \quad (3.34)$$

In the case of  $m_e < m_g$ , the leading contribution comes from the first term in the averaged equation of motion Eq. (3.15), while the second term is dominant for  $m_g < m_e$ .

The free-motion approximation is valid for orbits with small scattering angles. This means that the kinetic energy of the spatial motion should be sufficiently larger than that of the internal orientations. If vortices with non-zero charges have small relative velocities, the interaction Eq. (3.18) lasts for a long time interval  $\Delta t \approx 1/(m_g v)$  and the deviation of the trajectory from the free motion becomes large. Therefore  $\gamma_1$  and  $\gamma_2$  should be generically small so that

$$\gamma_1 \gamma_2 e^{-m_g a} \ll 1. \quad (3.35)$$

In Fig. 6-(a), we compare the scattering angles obtained by numerical calculations and those

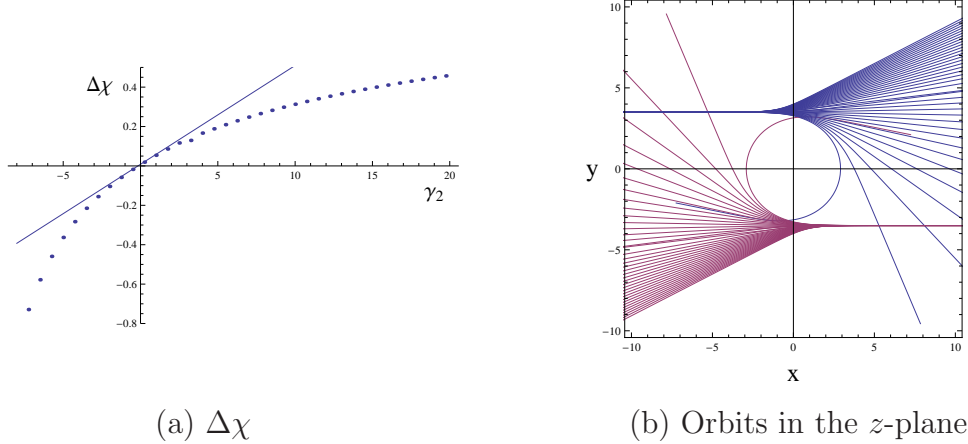


Fig. 6: (a) Scattering angles  $\Delta\chi$  as a function of  $\gamma_2$ , and (b) the corresponding orbits of the vortices in the  $z$ -plane ( $(x, y)$ -plane) for  $m_e = m_g = 1$ ,  $a = 7$ ,  $\vec{q}_1 = \vec{q}_2 = (1, 0, 0)$ ,  $\vec{n}_{10} = (0, 1, 0)$ ,  $\vec{n}_{20} = (0, 0, 1)$  and  $\gamma_1 = 3$ .

with the free-motion approximation. One finds that the free-motion approximation is indeed valid only for the scattering with small scattering angles. We also show the numerical results of the scattering with various initial conditions in Fig. 6-(b). Among them, there are orbits whose scattering angles exceed  $\pi/2$ . For such collisions, the free-motion approximation cannot be applied.

The terms with  $\gamma_I$  in the exponents (terms other than the first line) in Eq. (3.32) are contributions from the oscillating forces and become smaller for larger values of  $\gamma_I$ . Although those forces give the subdominant contributions, some of the subleading terms in Eq. (3.32) show resonant behaviors and become comparable to the leading term at  $\omega_1 = 0$ ,  $\omega_2 = 0$  or  $\omega_1 = \omega_2$ . This is because the corresponding forces in the equation of motion Eq. (3.12) do not oscillate for these values of frequencies, that is, the forces are not averaged in the case of vortices with synchronized orientations. In the case of  $m_e > m_g$  with  $\vec{q}_1 \cdot \vec{q}_2 = 0$ , the leading term Eq. (3.34) vanishes and the terms with the resonant behavior become dominant as shown in Fig. 7.

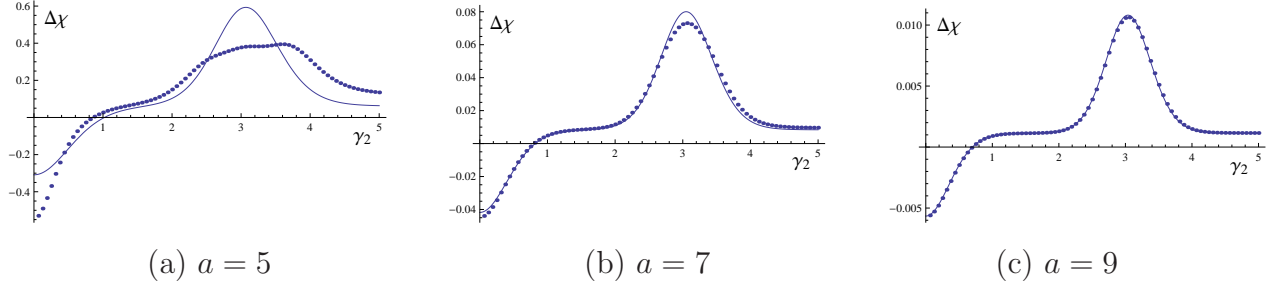


Fig. 7: Scattering angles as a function of  $\gamma_2$  for  $m_g = m_e = 1$ ,  $a = 5, 7, 9$ ,  $\gamma_1 = 3$ ,  $\vec{q}_1 = \vec{n}_{20} = (1, 0, 0)$  and  $\vec{q}_2 = \vec{n}_{10} = (0, 1, 0)$ . The numerical results and those of free-motion approximation are denoted by dotted lines and solid lines, respectively. The resonances can be seen at  $\gamma_2 = 0$  and  $\gamma_2 = \gamma_1 = 3$ .

### 3.4.2 Exchange of $Q$ -charges

In the scattering process, the charge of individual vortex in Eq. (3.6) is not conserved and its time dependence is given by the equation of motion for the orientation. Substituting the free motion Eq. (3.27) into the equation of motion Eq. (3.13) and integrating the right-hand side, we obtain the increment of the charge of the first vortex  $\Delta\vec{Q}_1$  ( $= -\Delta\vec{Q}_2$ ) as

$$\begin{aligned} \Delta\vec{Q}_1 = & -\frac{2\pi^2 c_g^2 m_g v}{g^2} \left( \gamma_1 \gamma_2 \vec{q}_1 \times \vec{q}_2 e^{-m_g a} + \gamma_2 \vec{q}_2 \times \vec{n}_{10} e^{-m_g a \sqrt{1+\gamma_1^2}} - \gamma_1 \vec{q}_1 \times \vec{n}_{20} e^{-m_g a \sqrt{1+\gamma_2^2}} \right) \\ & - \frac{2\pi^2 c_g^2 m_g v}{g^2} \frac{1 + 2\gamma_1 \gamma_2}{\sqrt{1 + (\gamma_1 + \gamma_2)^2}} \text{Re}[\vec{c}_1 \times \vec{c}_2] e^{-m_g a \sqrt{1 + (\gamma_1 + \gamma_2)^2}} \\ & - \frac{2\pi^2 c_g^2 m_g v}{g^2} \frac{1 - 2\gamma_1 \gamma_2}{\sqrt{1 + (\gamma_1 - \gamma_2)^2}} \text{Re}[\vec{c}_1 \times \vec{c}_2^*] e^{-m_g a \sqrt{1 + (\gamma_1 - \gamma_2)^2}}. \end{aligned} \quad (3.36)$$

Here, we have used the following integration formula

$$\begin{aligned} \int_{-\infty}^{\infty} dt \left( \frac{vt \pm ia}{\sqrt{(vt)^2 + a^2}} \right)^2 K_2(m \sqrt{(vt)^2 + a^2}) e^{i\omega t} \\ = \frac{2\pi}{(mv)^2} \left( \mp \omega - \frac{\frac{1}{2}(mv)^2 + \omega^2}{\sqrt{(mv)^2 + \omega^2}} \right) e^{-\frac{a}{v} \sqrt{(mv)^2 + \omega^2}}. \end{aligned} \quad (3.37)$$

For generic values of the angular velocities of the orientations  $\gamma_I$ , the leading contribution is given by

$$\Delta\vec{Q}_1 \approx -\frac{2\pi^2 c_g^2 m_g v}{g^2} \gamma_1 \gamma_2 \vec{q}_1 \times \vec{q}_2 e^{-m_g a}. \quad (3.38)$$

This contribution comes from the electric coupling Eq. (3.18) as in the case of  $\Delta\dot{z}_{12}$ . The resonant behavior of the subleading terms can also be seen at  $\gamma_1 = 0$ ,  $\gamma_2 = 0$  and  $\gamma_1 = \pm\gamma_2$ . In particular, if the vortices are initially uncharged ( $\gamma_1 = \gamma_2 = 0$ ), the increment  $\Delta\vec{Q}_1$  is given by

$$\Delta\vec{Q}_1 = -\frac{\pi^2 c_g^2 m_g v}{g^2} \vec{n}_{10} \times \vec{n}_{20} e^{-m_g a}. \quad (3.39)$$

Therefore, the vortices are charged after the scattering even if  $\vec{Q}_I = 0$  initially.

### 3.4.3 Exchange of energy

Let us now show that the energy can be transferred between two non-Abelian vortices through the scattering. This is another characteristic feature of the non-Abelian vortices. Since two Abelian vortices are identical objects without any internal degree of freedom, the energy transfer never occurs in the scattering of two Abelian vortices since only the elastic scattering is possible for two identical vortices.

The energy of an isolated vortex is given by the sum of the kinetic energies of the spatial motion and the orientation

$$E_I = \frac{\pi m_g^2}{g^2} |\dot{z}_I|^2 + \frac{g^2}{4\pi} |\vec{Q}_I|^2. \quad (3.40)$$

Assuming that  $\Delta \dot{z}_{12}$  and  $\Delta \vec{Q}_1$  are sufficiently small, we can calculate the total change in energy,  $\Delta E_1$  ( $= -\Delta E_2$ ), as

$$\begin{aligned} \Delta E_1 &\approx \frac{\pi m_g^2}{g^2} (\Delta \dot{z}_1 \dot{z}_1 + \dot{z}_1 \Delta \dot{z}_1) + \frac{g^2}{2\pi} \Delta \vec{Q}_1 \cdot \vec{Q}_1 \\ &\approx -2\pi E_0 c_g^2 \gamma_1 \gamma_2 (\vec{q}_1 \times \vec{q}_2) \cdot \left( \vec{n}_{10} e^{-m_g a \sqrt{1+\gamma_1^2}} + \vec{n}_{20} e^{-m_g a \sqrt{1+\gamma_2^2}} \right) \\ &\quad - 2\pi E_0 c_g^2 \frac{(\gamma_1 - \gamma_2)(1 + 2\gamma_1 \gamma_2)}{\sqrt{1 + (\gamma_1 + \gamma_2)^2}} \text{Im}[\vec{c}_1 \cdot \vec{c}_2] e^{-m_g a \sqrt{1+(\gamma_1+\gamma_2)^2}} \\ &\quad + 2\pi E_0 c_g^2 \frac{(\gamma_1 + \gamma_2)(1 - 2\gamma_1 \gamma_2)}{\sqrt{1 + (\gamma_1 - \gamma_2)^2}} \text{Im}[\vec{c}_1 \cdot \vec{c}_2^*] e^{-m_g a \sqrt{1+(\gamma_1-\gamma_2)^2}}, \end{aligned} \quad (3.41)$$

where  $E_0 \equiv \pi m_g^2 v^2 / (2g^2)$  is the kinetic energy associated to the initial relative velocity. Note that  $\dot{z}_1 = \frac{1}{2} \dot{z}_{12}$  in the center of mass frame. This shows that the energy is exchanged between two vortices only through the subleading interactions in general<sup>7</sup>, while the resonances dominate when  $\gamma_1 = \gamma_2$  and  $\gamma_1 = 0$  ( $\gamma_2 = 0$ ), as shown in Fig. 8.

### 3.5 Zero impact parameter

In this section, we discuss the vortex scattering with zero impact parameter. For simplicity, we consider the effective Lagrangian restricted to the subspace of the moduli space given by

$$z_{12} = r \in \mathbb{R}, \quad \beta_I = e^{i\varphi_I}. \quad (3.42)$$

This subspace is a fixed point set of the following reflection symmetry on the moduli space

$$z_{12} \rightarrow \bar{z}_{12}, \quad \beta_I \rightarrow \frac{1}{\bar{\beta}_I}. \quad (3.43)$$

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<sup>7</sup> The leading terms are of order  $\max\{e^{-m_e a}, e^{-m_g a}\}$ .



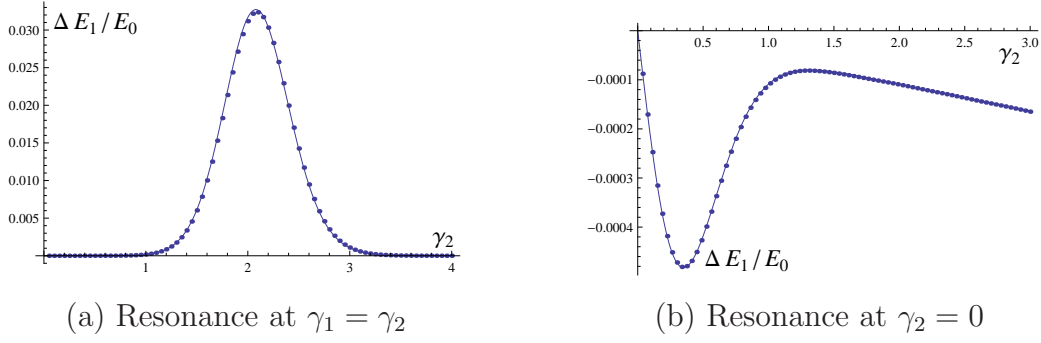


Fig. 8: The energy transfer  $\Delta E_1/E_0$  as a function of  $\gamma_2$  ( $m_g = m_e = 1$ ). (a)  $a = 9$ ,  $\gamma_1 = 2$ ,  $\vec{q}_1 = \vec{q}_2 = (1, 0, 0)$ ,  $\vec{n}_{10} = (0, 1, 0)$  and  $\vec{n}_{20} = (0, 0, 1)$ , (b)  $a = 9$ ,  $\gamma_1 = 1$ ,  $\vec{n}_{10} = \vec{n}_{20} = (0, 0, 1)$ ,  $\vec{q}_1 = (1, 0, 0)$  and  $\vec{q}_2 = (0, 1, 0)$ . The resonances can be seen at  $\gamma_2 = \gamma_1$  in (a) and at  $\gamma_2 = 0$  in (b).

According to the principle of symmetric criticality, any solution of the equation of motion restricted on such an invariant submanifold is automatically a stationary point of the original effective action. On this submanifold, the Lagrangian becomes

$$\begin{aligned}
L = & \frac{\pi}{2g^2} [1 - 2c_e^2 K_0(m_e r) - 2c_g^2 K_0(m_g r) \cos \varphi_r] m_g^2 \dot{r}^2 \\
& + \frac{\pi}{2g^2} [1 + c_g^2 K_0(m_g r)(1 + 3 \cos \varphi_r)] \dot{\varphi}_r^2 - \frac{2\pi}{g^2} c_g^2 K_1(m_g r) \sin \varphi_r m_g \dot{r} \dot{\varphi}_r \\
& + \frac{2\pi}{g^2} \left(1 - 2c_g^2 K_0(m_g r) \sin^2 \frac{\varphi_r}{2}\right) \dot{\varphi}_0^2,
\end{aligned} \tag{3.44}$$

where  $\varphi_0 = \frac{1}{2}(\varphi_1 + \varphi_2)$  and  $\varphi_r = \varphi_1 - \varphi_2$ . In this case, there exists only one non-vanishing conserved charge

$$Q = \frac{4\pi}{g^2} \left(1 - 2c_g^2 K_0(m_g r) \sin^2 \frac{\varphi_r}{2}\right) \dot{\varphi}_0, \tag{3.45}$$

which corresponds to the third component  $(\vec{Q})_3$  of the conserved Noether charge. By a Legendre transformation,  $\dot{\varphi}$  can be eliminated from the effective Lagrangian in favor of  $Q$ . As a result, the following potential is induced by the Noether charge

$$V_Q = \frac{g^2}{8\pi} Q^2 \left(1 - 2c_g^2 K_0(m_g r) \sin^2 \frac{\varphi_r}{2}\right)^{-1} \approx \frac{g^2}{8\pi} Q^2 \left(1 + 2c_g^2 K_0(m_g r) \sin^2 \frac{\varphi_r}{2}\right). \tag{3.46}$$

This potential is shown in Fig. 9. Then, the equations of motion are given by

$$\begin{aligned}
\ddot{r} = & - [c_e^2 m_e K_1(m_e r) + c_g^2 m_g K_1(m_g r) \cos \varphi_r] \dot{r}^2 \\
& - 2c_g^2 K_0(m_g r) \sin \varphi_r \dot{r} \dot{\varphi}_r + \frac{c_g^2}{m_g} K_1(m_g r) \sin^2 \frac{\varphi_r}{2} \left( \frac{g^4}{4\pi^2} Q^2 - \dot{\varphi}_r^2 \right),
\end{aligned} \tag{3.47}$$

$$\begin{aligned}
\ddot{\varphi}_r = & - c_g^2 m_g^2 K_2(m_g r) \sin \varphi_r \dot{r}^2 + c_g^2 m_g K_1(m_g r)(1 + 3 \cos \varphi_r) \dot{r} \dot{\varphi}_r \\
& - \frac{c_g^2}{2} K_0(m_g r) \sin \varphi_r \left( \frac{g^4}{4\pi^2} Q^2 - 3\dot{\varphi}_r^2 \right).
\end{aligned} \tag{3.48}$$

If the charge  $Q$  is sufficiently large compared to  $\dot{\varphi}_r, \dot{r}$  and the relative angle  $\varphi_r$  takes a generic value, two vortices recoil due to the potential  $V_Q$ , in which case we can continue to use the asymptotic metric. This phenomenon can be seen in the numerical calculation Fig. 4-(a). On the other hand, when  $Q$  is small or  $\varphi_r \approx 0$  ( $\vec{n}_1 \cdot \vec{n}_2 \approx 1$ ), the vortices can closely approach along the valley of the potential  $V_Q$ , in which case we can trace the dynamics until the asymptotic metric becomes invalid. Note that the cases with  $Q \approx 0$  and  $\varphi_r \approx \text{const.}$  correspond to the resonance at  $\gamma_1 = \gamma_2$ .

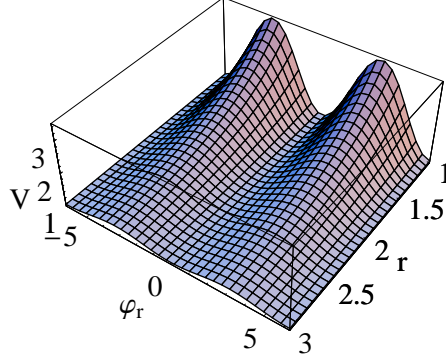


Fig. 9: The induced potential  $V_Q/(g^2 Q^2 / 8\pi)$  with  $c_e = c_g = 1.708$  and  $m_e = m_g = 1$ .

## 4 BPS Dyonic vortices

### 4.1 Dyonic vortices from a mass deformation

In this section we discuss the dynamics of BPS dyonic vortices in the  $U(2)$  gauge theory. In order to obtain the dyonic vortices, we add the adjoint scalar field  $\Sigma = \Sigma^0 t^a + \Sigma^a t^a$  and the following kinetic and mass terms [24] to the original Lagrangian Eq. (2.1)

$$\mathcal{L}_{\text{kin adj}} = \frac{1}{g^2} \text{Tr} [\mathcal{D}_\mu \Sigma \mathcal{D}^\mu \Sigma], \quad \mathcal{L}_{\text{mass}} = \text{Tr} [(\Sigma H - H M)(\Sigma H - H M)^\dagger], \quad (4.1)$$

where  $M$  is a 2-by-2 mass matrix. This is the only consistent mass deformation preserving all supersymmetry, when embedded into the supersymmetric theory with eight supercharges<sup>8</sup>. The mass matrix can be written as a linear combination of the Pauli matrices

$$M = \frac{1}{2} \vec{m} \cdot \vec{\sigma}. \quad (4.2)$$

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<sup>8</sup> Although the adjoint scalar  $\Sigma$  and the mass  $M$  can be triplets of  $SU(2) \subset SO(4)_R$  in (2+1)-dimensional  $\mathcal{N} = 4$  supersymmetric gauge theories, only one component of each triplets is relevant to the dyonic vortices.

In the following, we choose the mass to be  $\vec{m} = (0, 0, m)$ . This mass term induces the non-vanishing VEV for the adjoint scalar

$$\langle \Sigma \rangle = M, \quad (4.3)$$

and breaks the  $SU(2)_{C+F}$  symmetry to  $U(1)$  ( $U(1)_m$  from now on), for which the corresponding conserved charge is the component of  $\vec{Q}$  parallel to  $\vec{m}$ .

In this mass deformed model, the BPS bound for the energy density is given by [23, 41]

$$\mathcal{E} \geq -c f_{12} - i\epsilon^{ij} \partial_i \text{Tr} [\mathcal{D}_j H H^\dagger] + i \text{Tr} [H M \mathcal{D}_0 H^\dagger - \mathcal{D}_0 H M H^\dagger] + \frac{2}{g^2} \partial_i \text{Tr} [F_{0i} \Sigma]. \quad (4.4)$$

The first two terms are the energy density of vortices and the third term is the conserved Noether charge for the unbroken  $U(1)_m$  symmetry. The forth term is the divergence of the the electric flux in the internal direction specified by the adjoint scalar  $\Sigma$ . We call this quantity simply “electric charge density”. The BPS bound is saturated if Eq. (2.9) and the following equations are satisfied

$$\mathcal{D}_0 H = -i(\Sigma H - H M), \quad \mathcal{D}_i \Sigma = F_{0i}, \quad \mathcal{D}_0 \Sigma = 0, \quad (4.5)$$

$$\mathcal{D}_i \left[ \frac{2}{e^2} f_{0i} t^0 + \frac{2}{g^2} F_{0i}^a t^a \right] = -i(H \mathcal{D}_0 H^\dagger - \mathcal{D}_0 H H^\dagger). \quad (4.6)$$

These BPS equations for dyonic vortices can be simplified by choosing the gauge in which the time component of the gauge field takes the form

$$W_0 = M - \Sigma. \quad (4.7)$$

Then, Eqs. (4.5), (4.6) become

$$\partial_0 H = i[H, M], \quad \partial_0 W_i = i[W_i, M], \quad \partial_0 \Sigma = i[\Sigma, M], \quad (4.8)$$

$$\mathcal{D}_i \mathcal{D}_i \left[ \frac{2}{e^2} \Sigma^0 t^0 + \frac{2}{g^2} \Sigma^a t^a \right] - \{H H^\dagger, \Sigma\} = -2H M H^\dagger. \quad (4.9)$$

From these equations we find that the BPS dyonic vortex solutions can be obtained by solving Eq. (4.9) with respect to  $\Sigma$  in a static vortex background satisfying Eq. (2.9) and then rotating the orientation as

$$H \rightarrow U^\dagger H U, \quad W_i \rightarrow U^\dagger W_i U, \quad \Sigma \rightarrow U^\dagger \Sigma U, \quad U = e^{iMt} \in U(1)_m. \quad (4.10)$$

Therefore, the dyonic vortex solutions are stable stationary configurations with rotating orientation. An important fact is that the no static force is exerted among the BPS dyonic vortices, that is, the electrostatic interaction Eq. (3.18) is canceled by another interaction induced by the adjoint scalar  $\Sigma$ .

One can find the energy of the BPS saturated configurations by integrating the right-hand side of Eq. (4.4) as

$$E = 2\pi ck + \vec{m} \cdot \vec{Q}, \quad (4.11)$$

where we have used

$$\vec{m} \cdot \vec{Q} = i \int d^2x \text{Tr} [H M \mathcal{D}_0 H^\dagger - \mathcal{D}_0 H M H^\dagger]. \quad (4.12)$$

Note that the forth term in Eq. (4.4), as well as the second term, has no contribution to the total energy since the electric field is screened and decays exponentially in the Higgs phase. Note that the dyonic vortices have non-trivial charge distributions (see Fig. 10), although they have no total electric charge.

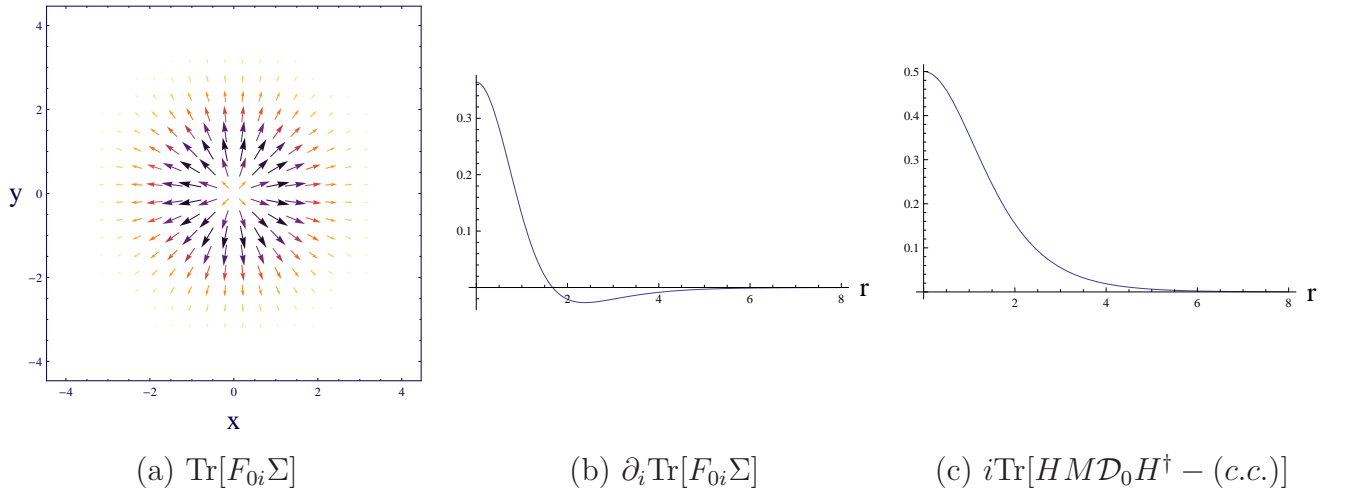


Fig. 10: An example of a single dyonic vortex configuration (numerical solution) for  $m_g = m_e = m = 1$ ,  $\theta_0 = \frac{\pi}{2}$ . (a) The electric flux radially spreads from the vortex center and decays exponentially. (b) The electric charge density is positive inside the vortex core while it is negative around the core. (c) The conserved Noether charge density is positive everywhere and the total charge is non-zero.

We can also discuss the dyonic configurations in the effective theory of vortices. The deformation term Eq. (4.1) induces the following potential on the moduli space

$$V_m = g_{i\bar{j}} k^i \bar{k}^j, \quad k^i \equiv \vec{m} \cdot \vec{\xi}^i, \quad (4.13)$$

where  $\vec{\xi}^i$  is the holomorphic Killing vector for  $SU(2)$  defined in Eq. (3.5) and  $k^i$  is that for  $U(1)_m$ . In this deformed effective theory of vortices, the energy is bounded by the Noether charge  $Q$

$$E = g_{i\bar{j}} \left( \dot{\phi}^i \dot{\phi}^{\bar{j}} + k^i \bar{k}^j \right) = g_{i\bar{j}} \left[ \left( \dot{\phi}^i \mp k^i \right) \overline{\left( \dot{\phi}^j \mp k^j \right)} \pm \left( \dot{\phi}^i \bar{k}^j + \dot{\phi}^j k^i \right) \right] \geq \pm \vec{m} \cdot \vec{Q}. \quad (4.14)$$

Therefore the solutions of the dyonic vortices, which are trajectories on the moduli space with minimum energy for a given value of  $\vec{m} \cdot \vec{Q}$ , are determined from the BPS equation

$$\dot{\phi}^i = \pm k^i. \quad (4.15)$$

In the rest of this paper, we take the positive sign for BPS dyonic vortices. In the case of  $\vec{m} = (0, 0, m)$ , the dyonic vortex solution is given by

$$z_I = z_{I0}, \quad \beta_I = \beta_{I0} e^{imt}, \quad \left( \cdot \cdot \quad k^i \frac{\partial}{\partial \phi^i} = im \sum_{I=1}^k \beta_I \frac{\partial}{\partial \beta_I} \right), \quad (4.16)$$

where  $z_{I0}$  and  $\beta_{I0}$  are complex constants which are related to the Noether charge by

$$\vec{m} \cdot \vec{Q} = g_{i\bar{j}} (\dot{\phi}^i \bar{k}^j + \dot{\bar{\phi}}^j k^i) = 2g_{i\bar{j}} k^i \bar{k}^j, \quad (4.17)$$

where we have used Eq. (4.15). Again, we find that the orientations of the vortices are rotating with the same period.

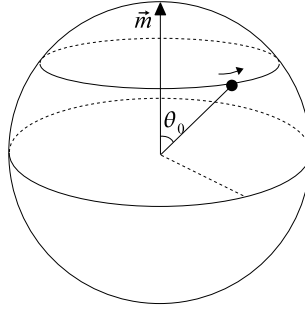


Fig. 11: The trajectory of the internal orientation for a single dyonic vortex. The orientation rotates around a small circle at a fixed latitude  $\theta_0$ .

As an example let us consider the case of a single dyonic vortex. Since the dynamics of the vortex position is trivial in this case, we consider only the vortex orientation. For a single vortex, the mass deformed effective Lagrangian can be written in terms of the spherical coordinates  $\beta = \tan \frac{\theta}{2} e^{i\varphi}$  as

$$L = \frac{\pi}{g^2} \left[ \dot{\theta}^2 + \sin^2 \theta (\dot{\varphi}^2 - m^2) \right]. \quad (4.18)$$

The Noether charge  $Q \equiv \vec{m} \cdot \vec{Q} / |\vec{m}|$  in this case is the conjugate momentum of  $\varphi$

$$Q = \frac{\partial L}{\partial \dot{\varphi}} = \frac{2\pi}{g^2} \sin^2 \theta \dot{\varphi}. \quad (4.19)$$

The dyonic vortex solution is given by

$$\theta = \theta_0, \quad \varphi = mt + \varphi_0, \quad Q = \frac{2\pi m}{g^2} \sin^2 \theta_0. \quad (4.20)$$

where  $\theta_0$  and  $\varphi_0$  are constants. The trajectory corresponding to the dyonic vortex is a small circle at  $\theta_0$ , on which the following effective potential is minimized

$$V(\theta) = V_Q + V_m = \frac{g^2}{4\pi \sin^2 \theta} Q^2 + \frac{\pi \sin^2 \theta}{g^2} m^2 \geq \frac{2\pi m^2}{g^2} \sin^2 \theta_0, \quad (4.21)$$

where  $V_Q$  is the potential induced by the Noether charge  $Q$ . Let us consider a small fluctuation  $\delta\theta$  from the trajectory (4.20) corresponding to the dyonic vortex:  $\theta = \theta_0 + \delta\theta$ . The effective potential for  $\delta\theta$  is given by

$$V(\theta) \approx \frac{2\pi m^2}{g^2} \sin^2 \theta_0 + \frac{4\pi m^2}{g^2} \cos^2 \theta_0 \delta\theta^2. \quad (4.22)$$

Therefore, if the dyonic vortex is excited, the angle parameter  $\theta$  oscillates with frequency  $\omega = 2m \cos \theta_0$ , which is generically of order  $m = |\vec{m}|$ . Thus we find that there exist massive modes around the dyonic vortex configuration.

## 4.2 Dynamics of dyonic vortices

In this section we discuss the dynamics of two dyonic vortices by using the equations of motion Eqs. (3.12) and (3.13) modified by the mass term. Since the action of the Killing vector on the moduli parameters is given by

$$(\vec{m} \cdot \vec{\xi} + \vec{m} \cdot \vec{\xi}^*) z_I = 0, \quad (\vec{m} \cdot \vec{\xi} + \vec{m} \cdot \vec{\xi}^*) \vec{n}_I = \vec{m} \times \vec{n}_I, \quad (4.23)$$

the Killing potential  $V_m$  can be obtained from the kinetic terms of the effective Lagrangian by dropping  $\dot{z}_I$  and replacing  $\dot{\vec{n}}_I$  as

$$\dot{\vec{n}}_I \rightarrow \vec{m} \times \vec{n}_I. \quad (4.24)$$

The full mass deformed Lagrangian can be found in Appendix A. Then the BPS equations for dyonic vortices can be rewritten as

$$\dot{\vec{n}}_I = \vec{m} \times \vec{n}_I. \quad (4.25)$$

The BPS solution takes the form

$$\vec{n}_I = \frac{1}{m} \left[ \sqrt{1 - |\vec{v}_I|^2} \vec{m} + \cos(mt) m \vec{v}_I + \sin(mt) \vec{m} \times \vec{v}_I \right], \quad (4.26)$$

where  $\vec{v}_I$  are vectors such that  $\vec{v}_I \cdot \vec{m} = 0$ .

Let us next see how the equations of motion for the relative position and the orientations are modified. The mass deformation changes  $\vec{\alpha}_1 \cdot \vec{\alpha}_2$  in the last term of the equation of motion Eq. (3.12) as

$$\vec{\alpha}_1 \cdot \vec{\alpha}_2 \rightarrow \vec{\alpha}_1 \cdot \vec{\alpha}_2 - \vec{\alpha}'_1 \cdot \vec{\alpha}'_2, \quad (4.27)$$

where  $\vec{\alpha}'_I$  is the vectors obtained from  $\vec{\alpha}_I = \dot{\vec{n}}_I - i\vec{n}_I \times \dot{\vec{n}}_I$  by replacing  $\dot{\vec{n}}_I$  with  $\vec{m} \times \vec{n}_I$

$$\vec{\alpha}'_I \equiv \vec{m} \times \vec{n}_I - i\vec{n}_I \times (\vec{m} \times \vec{n}_I). \quad (4.28)$$

Therefore, if the pair of the vortices are near BPS  $\dot{\vec{n}}_I \approx \vec{m} \times \vec{n}_I$ , the contributions from the last term of Eq. (3.12) and the mass deformation are small

$$\vec{\alpha}_1 \cdot \vec{\alpha}_2 - \vec{\alpha}'_1 \cdot \vec{\alpha}'_2 \approx 0. \quad (4.29)$$

Similarly, we can also show that there exists a similar cancellation in the equation of motion for the orientation Eq. (3.13)

$$\vec{\alpha}_1(\vec{n}_2 \cdot \vec{\alpha}_1 + 2\vec{n}_1 \cdot \vec{\alpha}_2) - \vec{\alpha}'_1(\vec{n}_2 \cdot \vec{\alpha}'_1 + 2\vec{n}_1 \cdot \vec{\alpha}'_2) \approx 0. \quad (4.30)$$

In the following, we will see that due to these cancellation, the behavior of the dyonic vortices are quite different from that of vortices before the mass deformation.

#### 4.2.1 Angular momentum, Lorentz Force and a Bound State

Let us first discuss the case in which the relative velocity of the dyonic vortices are small. By using the effective Lagrangian Eq. (A.1), we can write down the conserved angular momentum in terms of the radial coordinates  $z_{12} = re^{i\chi}$  as

$$\mathbf{L}_\chi = \frac{\partial L}{\partial \dot{\chi}} \approx \frac{\pi m_g^2}{g^2} r^2 \dot{\chi} + A_\chi(r), \quad (4.31)$$

where we have neglected a small term proportional to  $\dot{\chi} K_0$ . The function  $A_\chi$  is given by

$$A_\chi(r) = - \left( \vec{n}_1 \cdot \vec{Q}_2 + \vec{n}_2 \cdot \vec{Q}_1 \right) c_g^2 K_1(m_g r) m_g r. \quad (4.32)$$

The first term in Eq. (4.31) is the ordinary angular momentum of a free particle and the additional term  $A_\chi(r)$  can be interpreted as a “gauge potential” generated by the motion of the orientational moduli.

In the massless case discussed in section 3, the contribution of  $A_\chi(r)$  is small compared with the leading terms. This can be seen by averaging over the rapid motions of the orientations. Since they rotate around great circles of  $\mathbb{CP}^1$ , it follows that

$$\left\langle \vec{n}_1 \cdot \vec{Q}_2 + \vec{n}_2 \cdot \vec{Q}_1 \right\rangle = 0. \quad (4.33)$$

The mass deformation drastically changes this situation. Since the orientations rotate around small circles in the case of the dyonic vortices, the average of the gauge potential is non-zero

$$\left\langle \vec{n}_1 \cdot \vec{Q}_2 + \vec{n}_2 \cdot \vec{Q}_1 \right\rangle = \frac{2\pi}{g^2} \vec{m} \cdot (\vec{n}_1 + \vec{n}_2)(1 - \vec{n}_1 \cdot \vec{n}_2). \quad (4.34)$$

Note that  $\vec{m} \cdot \vec{n}_I$  and  $\vec{n}_1 \cdot \vec{n}_2$  are independent of time for BPS dyonic vortices and slowly vary for an interacting pair of vortices. Due to the contribution from the gauge potential  $A_\chi$ , the motions of the dyonic vortices with small relative velocity  $v = |\dot{z}_{12}|$  becomes quite different from that without the mass deformation. For a pair of near BPS dyonic vortices  $\dot{\vec{n}}_I \approx \vec{m} \times \vec{n}_I$ , the non-trivial gauge potential  $A_\chi(r)$  gives the dominant contribution via the Lorentz force

$$\frac{\pi m_g^2}{g^2} \ddot{z}_{12} \approx -iB \dot{z}_{12}, \quad B(r) \equiv \frac{1}{r} \frac{dA_\chi(r)}{dr}. \quad (4.35)$$

Assuming that the relative velocity  $v = |\dot{z}_{12}|$  and the massive oscillations  $[\delta\theta$  in Eq. (4.22)] are small, we can write down the energy of this system as

$$E \approx \frac{\pi m_g^2}{2g^2} \dot{r}^2 + V_L(r), \quad V_L(r) = \frac{g^2}{\pi} \left( \frac{\mathbf{L}_\chi - A_\chi(r)}{m_g r} \right)^2, \quad (4.36)$$

where we have neglected irrelevant terms. For an arbitrary relative distance  $r = r_0$ , we can always adjust  $\mathbf{L}_\chi$  so that  $V_L = 0$ , namely

$$\mathbf{L}_\chi = A_\chi(r_0). \quad (4.37)$$

This minimum energy configuration corresponds to the BPS solution, for which the velocity of the relative position is zero  $\dot{r} = \dot{\chi} = 0$ . As shown in Fig. 12-(a), this is a stable point of the positive semi-definite potential  $V_L$ , and thus the relative distance oscillates around the minimum for a sufficiently small excitation energy. The time dependence of the relative angle  $\chi$  can be determined from Eq. (4.31). As a result, we can find that the dyonic vortices drift along a contour

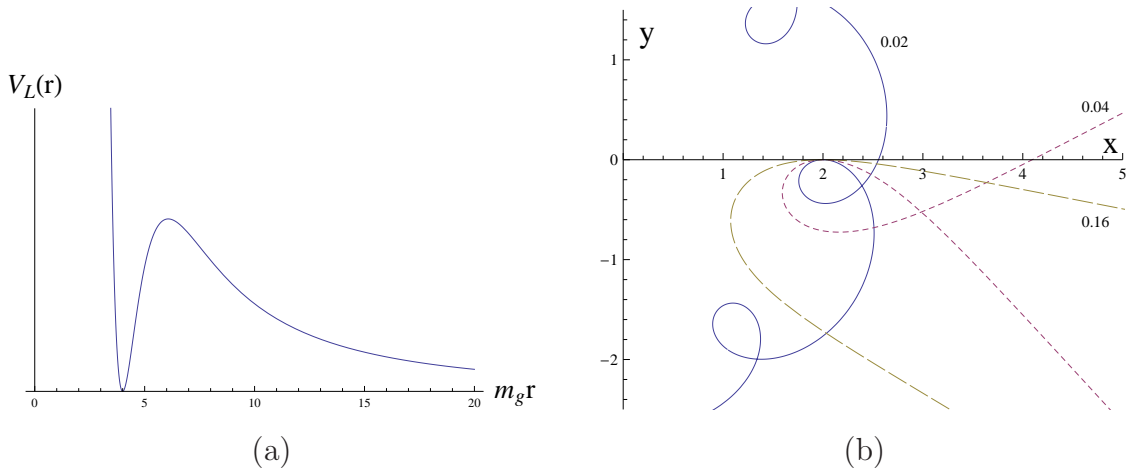


Fig. 12: (a) The potential  $V_L(r)$  representing the Lorentz force for  $m_g r_0 = 4$ . (b) Examples of orbits (numerical solutions) for  $m_g v / |\vec{m}| = 0.02$  (solid line), 0.04 (dotted line), 0.16 (dashed line). Solid line shows an extended coil winding circularly around the origin.

line of the magnetic field similarly to a charged particle in a background magnetic field. The



orbit of each dyonic vortex in the  $z$ -plane takes the form of an extended coil winding around a contour line (see Fig.13-(a)). In this sense, a pair of dyonic vortices can form a “bound state” with a large relative distance  $r_0 \gg m_g^{-1}$ . The frequency  $\omega_L$  of the oscillation around the stable point  $r = r_0$  is given by

$$\omega_L \approx \frac{g^2}{\pi m_g^2} B(r_0) \propto K_0(m_g r_0), \quad (4.38)$$

and is generically quite small  $|\omega_L| \ll |\vec{m}|$  for well-separated vortices with  $m_g r_0 \gg 1$ . The radius of the coil  $r_{\text{coil}}$ , the velocity of the drift  $v_D$  and the period of the large circular motion  $T_B$  are respectively estimated as

$$r_{\text{coil}} \approx \frac{v}{|\omega_L|}, \quad v_D \approx \frac{m_g v^2}{2|\omega_L|}, \quad T_B \approx \frac{4\pi|\omega_L|r_0}{m_g v^2}, \quad (4.39)$$

where we have assumed that the relative velocity  $v = |\dot{z}_{12}|$  is sufficiently small. Note that the dyonic vortices with  $v$  larger than a certain critical value run away to infinity as illustrated in Fig.12-(b).

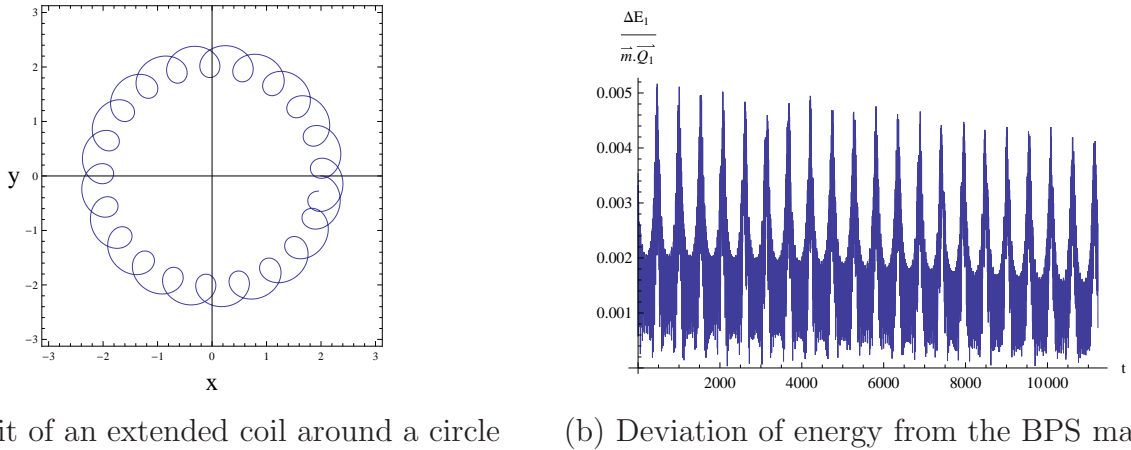


Fig. 13: An example of the bound state (numerical solution) for  $m_g v / |\vec{m}| = 0.016$ . (a) The orbit in  $z$ -plane. (b) The deviation of energy  $\Delta E_1$  (divided by the BPS mass) as a function of time.

In this analysis, we have neglected subleading terms in the equation of motion. This is valid if the excitation energy is sufficiently small since the subleading terms just change the potential  $V_L(r)$  slightly for a small relative velocity  $v$ . We have also assumed that the motions of the orientations are kept near BPS  $\dot{n}_I \approx \vec{m} \times \vec{n}_I$  and fluctuations of the massive modes  $\delta\theta_I$  are small. These assumptions are justified if we take the near BPS initial condition with sufficiently small  $v$ , since the geodesic force which is independent of  $\dot{z}_{12}$  is almost cancelled by the potential term induced by the mass deformation. Therefore, we expect that the bound state is stable as long as the deviation of the energy of each vortex is sufficiently small. Fig.13-(b) shows an example of the deviation of the energy from the BPS mass during the period  $T_B$ .

Now let us discuss the origin of the Lorentz force Eq. (4.35). Since dyonic vortices have both magnetic and electric fluxes, their asymptotic interaction can be well described by regarding them as point-like sources with both magnetic and electric moments. The static electric-electric and magnetic-magnetic interactions are canceled by the other scalar fields, so that the dominant interaction for a small relative velocity is the electric-magnetic interaction which is proportional to  $\dot{z}_{12}$ . Therefore, each slowly moving dyonic vortex feels the Lorentz force of the form Eq. (4.35) from the fluxes of the other dyonic vortex.

#### 4.2.2 Scattering of dyonic vortices

In this section, we discuss the scattering of dyonic vortices by using the free motion approximation as in section 3. For simplicity, we restrict the initial conditions for the vortices to the BPS states, that is

$$z_{12} = vt + ia, \quad \vec{n}_I = \frac{1}{m} \left[ \sqrt{1 - |\vec{v}_I|^2} \vec{m} + \cos(mt) \vec{v}_I + \sin(mt) \vec{m} \times \vec{v}_I \right]. \quad (4.40)$$

Substituting this into the equation of motion Eq. (3.12) modified by the mass deformation Eq. (4.27), we obtain the following equation for the relative position  $z_{12}$

$$\begin{aligned} \ddot{z}_{12} = & -m_e v^2 c_e^2 \frac{vt - ia}{\sqrt{(vt)^2 + a^2}} K_1(m_e \sqrt{(vt)^2 + a^2}) \\ & - m_g v^2 c_g^2 \frac{vt - ia}{\sqrt{(vt)^2 + a^2}} K_1(m_g \sqrt{(vt)^2 + a^2}) \vec{n}_1 \cdot \vec{n}_2 \\ & - 2vic_g^2 K_0(m_g \sqrt{(vt)^2 + a^2}) \vec{m} \cdot (\vec{n}_1 + \vec{n}_2)(1 - \vec{n}_1 \cdot \vec{n}_2). \end{aligned} \quad (4.41)$$

Here  $\vec{n}_1 \cdot \vec{n}_2$  and  $\vec{m} \cdot \vec{n}_I$  are constant since we have assumed that the orientation modes satisfy the BPS equations  $\dot{\vec{n}}_I = \vec{m} \times \vec{n}_I$ . Note that the forces vanish in the limit  $v \rightarrow 0$  due to the BPS properties of the dyonic vortices. The total change in the relative velocity  $\Delta \dot{z}_{12}$  is

$$\frac{\Delta \dot{z}_{12}}{2\pi i v} = \frac{1}{2} c_e^2 e^{-m_e a} + \frac{1}{2} c_g^2 e^{-m_g a} \vec{n}_1 \cdot \vec{n}_2 - c_g^2 e^{-m_g a} \frac{\vec{m} \cdot (\vec{n}_1 + \vec{n}_2)}{m_g v} (1 - \vec{n}_1 \cdot \vec{n}_2). \quad (4.42)$$

Similarly, we can calculate the total change in the  $Q$ -charge

$$\Delta(\vec{m} \cdot \vec{Q}_1) = -\Delta(\vec{m} \cdot \vec{Q}_2) = -\frac{\pi c_g^2 m_g v}{2} (\vec{n}_1 \times \vec{n}_2) \cdot \vec{m} \left[ 1 + \frac{2\vec{m} \cdot (\vec{n}_1 + \vec{n}_2)}{m_g v} \right] e^{-m_g a}. \quad (4.43)$$

From these results, we find that  $\Delta \dot{z}_{12}$  and  $\Delta(\vec{m} \cdot \vec{Q}_I)$  do not vanish even for small relative velocity<sup>9</sup>  $v \approx 0$ . This is because the Lorentz force (the last term in Eq. (4.41)) is proportional to  $v$  and its integrated effect is independent of  $v$ . Before the mass deformation, the differences  $\Delta \dot{z}_{12}$  and  $\Delta \vec{Q}_I$

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<sup>9</sup> Although the free motion approximation would not be a good approximation for  $v \approx 0$ , the results Eqs. (4.42) and (4.43) are qualitatively correct if the impact parameter  $a$  is sufficiently large.

vanish for a pair of slowly moving BPS vortices ( $v \rightarrow 0$ ,  $\gamma_I = 0$ ) as we can see from Eqs. (3.33) and (3.36). Note that in both cases the interaction vanishes in the limit  $v \rightarrow 0$  because of the BPS properties. This is one of the typical differences between BPS vortices before and after the mass deformation.

Since the first two terms in Eq. (4.42) can be obtained by taking the limit  $\gamma_{1,2} \rightarrow 0$  in Eq. (3.32), they consistently reduce to Eq. (3.32) in the limit  $|\vec{m}| \rightarrow 0$ . Note that  $\gamma_I \propto |\vec{Q}_I|$  are of order  $|\vec{m}|$  for the BPS dyonic vortices. The effect of the mass deformation lies in the last term, which becomes dominant if  $m_g v \ll |\vec{m}|$ . On the other hand, the free-motion approximation gives quantitatively precise results when the relative velocity  $v$  is sufficiently large :  $m_g v \gg m e^{-m_g a}$ . In such situations<sup>10</sup>, we can show that the impact parameter  $a$  should be sufficiently larger than the minimum of the potential  $r_0$  which is given by

$$|A_\chi(r_0)| = |\mathbf{L}_\chi| = \frac{\pi}{g^2}(m_g a)(m_g v). \quad (4.44)$$

Therefore Eqs. (4.42) and (4.43) are valid only for the scattering of dyonic vortices repelled by the outer barrier of the potential  $V_L(r)$ .

As we can see from Eqs. (4.42) and (4.43), there is no contribution of order  $m^2 \propto (\gamma_I)^2$ . This is one of the most striking features of the dyonic vortices, which makes the Lorentz force dominant and ensures the stability of the bound states. Strictly speaking, we have to consider contributions from the massive oscillations which we have ignored for simplicity. We expect that massive modes are irrelevant for the near BPS configurations, while they would cause instabilities of highly excited bound states<sup>11</sup>.

## 5 Conclusion and Discussion

We have studied the dynamics of non-Abelian vortices by the moduli space approximation using the asymptotic metric of well-separated non-Abelian vortices in the  $U(N)$  gauge theory with  $N$  Higgs scalar fields in the fundamental representation. Since non-Abelian vortices carry the orientational moduli  $\mathbb{C}P^{N-1}$  individually, they can have Noether charges as internal momentum of them, whereas ANO vortices in the Abelian-Higgs model have no such charges.

We have found that the vortices with the same charges repel while those with the opposite charges attract. We have shown that the charges of vortices can change during the scattering process with the total energy conserved; the kinetic energy of orientational moduli can be transferred

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<sup>10</sup> In the opposite case  $m_g v \gg |\vec{m}|$ , the approximation is always valid, but the effects of the mass deformation become subleading. For instance, there is no minimum in the potential  $V_L(r)$  in such a case.

<sup>11</sup> In the latter case, there would be corrections with resonant behaviors in the scattering angle  $\Delta\dot{z}_{12}$ , etc.

to the kinetic energy in real space and vice versa. As results, we have found that in scattering of two non-Abelian vortices, i) the scattering angle depends on the internal orientation, especially parallel orientations give repulsion while anti-parallel orientations give attraction, ii) the energy of real and internal spaces can be transferred, iii) the energy and charge transfer between two vortices occur, and iv) some resonances appear due to synchronization of the orientations.

By introducing the mass deformation into the original theory, the color-flavor symmetry  $SU(N)$  is explicitly broken to  $U(1)^{N-1}$ . Noether charges of these  $U(1)^{N-1}$  can be actually regarded as  $U(1)^{N-1}$  gauge charges where the Cartan subgroup of  $U(N)$  survives in the low energy theory in the large mass limit. We have shown that the dominance of the Lorentz force between vortices gives a bound state with a coiling orbit.

Here we address several discussions.

One may question what are the conditions for the existence of bound states in more general cases. To form a bound state, therefore, vortices are needed only to have

1. a magnetic flux (a topological charge) as the definition,
2. a sufficiently large conserved charge,
3. a sufficiently small static force.

Since large conserved charges generically induce large interactions between vortices, we need to prepare a special system for cancellation of static forces like a BPS state. Even in the Abelian case, vortices can have conserved charges. For instance, Abelian semi-local vortices [45, 46] can acquire  $Q$ -charges after the mass deformation. In the strong coupling limit  $e \rightarrow \infty$ , they reduce to  $Q$ -lumps and their interactions have been studied [19, 47].

As future problems, the followings might be interesting.

Our study in this paper has been based on the moduli space metric for well-separated vortices and therefore we cannot discuss vortices with a small separation. One question is what the fate of a pair of vortices with anti-parallel charges is. Will they form a bound state with a very small separation? Or will they reduce to a composite state of coincident vortices (singlet or triplet) [35, 38] after radiations of massive particles? The head-on collision of two vortices was studied previously based on the metric on the moduli subspace of two coincident vortices (at zero distance) [39]. However the moduli space metric for two vortices at arbitrary distance is not known, which is needed to answer the above questions. Also, beyond the moduli space approximation, one has to study dynamics by numerically solving the original equations of motion.

We have found that energy transfer and Noether charge transfer between two scattering vortices. This property may support the Boltzman's principle of equality when one considers statistical mechanics of many vortices. Partition function of a gas of non-Abelian vortices was calculated in [48] with assuming the principle of equality so that the calculation is reduced to the integration over the moduli space of vortices. Therefore, one important question is if a large number of scattering of vortices leads to the ergodic theorem so that one can assume the Boltzman's principle of equality.

Non-Abelian vortices are called semi-local if there are more flavors than the number of color. Non-Abelian semi-local vortices have size moduli and they reduces to local vortices when the size modulus is sent to zero. One of characteristic properties of semi-local vortices is that their profile functions decay polynomially but not exponentially as local vortices. Consequently the size modulus is non-normalizable. Moreover the  $\mathbb{C}P^{N-1}$  orientational moduli of a single non-Abelian semi-local vortex are also non-normalizable [49] when the size modulus is non-zero, but they are normalizable only when the size modulus is zero [50]. No metric can be defined for non-normalizable moduli, and dynamics cannot be discussed for those. However we can discuss the dynamics of two semi-local vortices because relative orientations and relative size between them are normalizable. In fact dynamics such as head-on collision was studied for Abelian [46] and non-Abelian [39] semi-local vortices. Since semi-local vortices can be approximated by lump solutions at large distance compared with Compton wave length of massive particles, the dynamics of semi-local vortices at large distance can be approximated by that of lumps [19].

One interesting generalization is changing geometry from a flat space to geometry with non-trivial cycles such as a cylinder [54], a torus [48], and Riemann surfaces with higher genus [55]. For semi-local vortices in compact Riemann surfaces, there is no problem of the non-normalizability and we can discuss their dynamics without introducing any cut-off scale by hand. In fact the moduli space metric was found for well-separated non-Abelian vortices on Riemann surfaces [56]. However dynamics during a long time will be difficult using the asymptotic metric because distances between vortices cannot be kept large for compact spaces.

Another interesting extension is changing gauge groups. Non-Abelian vortices were extended to arbitrary gauge groups  $G$  in the form of  $(U(1) \times G)/C(G)$  with the center  $C(G)$  of  $G$  [51]. Especially the cases of  $G = SO(N), USp(2N)$  have been studied in detail [52, 53]. However they are semi-local vortices in general [52] so that dynamics at large separation is the one of lumps [19].

We have studied dynamics of BPS vortices. There exist static force between non-BPS vortices [57]. Superconductors are classified into two types, type I and type II, depending on whether the static force between two vortices is attractive(type I) or repulsive (type II). A classification

of non-Abelian superconductors will be more complicated due to the existence of charges of non-BPS non-Abelian vortices [58]. When the couplings are close to the critical coupling (near BPS), dynamics of non-BPS vortices can be studied by the moduli space approximation plus a potential for the static forces, which remains as an important future problem. Inclusion of Chern-Simons terms [59] is also possible extension.

Our studies have been restricted to particle dynamics, namely vortices in 2+1 dimensions. In 3+1 dimensions, vortices are (cosmic) strings which have one spatial dimension in their world-sheet. The moduli space approximation can be applied when the angle between two cosmic strings is small. Collision of two non-Abelian cosmic strings was studied based on the metric on the moduli subspace of two coincident vortices [60, 39]. Especially it was found in [39] that orientational moduli of two non-Abelian vortices must be aligned and scatter with 90 degree angle, when they collide in head on, except for a fine-tuned collision. This implies that two non-Abelian cosmic strings reconnect each other. This result was obtained just before and after the collision moment in the linear order in time. On the other hand, Eq. (3.26) shows that orientational moduli of two vortices in head-on collision tend to be aligned. It suggests that two non-Abelian cosmic strings consistently reconnect each other as a long time behavior by feeling attraction between two orientational moduli like a ferromagnet. However we also have found in this paper that two vortices repel when charges induced by the motion of internal orientational moduli of two vortices are opposite. In this case, we expect that two cosmic strings merely reconnect each other.

We have found many new feature of dynamics or scattering of non-Abelian vortices, which are based on the existence of non-Abelian internal moduli of individual solitons. Therefore the similar properties should hold for scattering of other kinds of non-Abelian solitons such as Yang-Mills instantons, non-Abelian monopoles [30], and non-Abelian kinks [32].

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## A The action in terms of the orientation vectors $\vec{n}_I$

In the case of  $U(2)$  gauge theory, it is convenient to describe orientations of vortices in terms of three-component unit vectors  $\vec{n}_I$ . The Kähler potential in Eqs. (2.17) and (2.18) give the effective Lagrangian

$$L = L_{\text{kin}} + L_{\text{int}} - V_m. \quad (\text{A.1})$$

Introducing Lagrange multipliers  $\lambda_I \in \mathbb{R}$ , the effective Lagrangian  $\mathcal{L}_{\text{eff}}$  can be rewritten in terms of  $\vec{n}_I$ . The kinetic terms giving free motions are obtained as

$$L_{\text{kin}} = \sum_I \frac{2\pi}{g^2} \left( \frac{1}{2} m_g^2 |\dot{z}_I|^2 + \frac{1}{2} |\dot{\vec{n}}_I|^2 + \lambda_I (|\vec{n}_I|^2 - 1) \right). \quad (\text{A.2})$$

Note that the quantity  $\Theta_{12}$  in the Kähler potential is  $\Theta_{12} = \vec{n}_1 \cdot \vec{n}_2$  and

$$d\beta \frac{\partial}{\partial \beta} = \frac{1}{2} \left( d\vec{n} - \frac{\vec{n}}{|\vec{n}|^2} (\vec{n} \cdot d\vec{n}) - i\vec{n} \times d\vec{n} \right) \cdot \frac{\partial}{\partial \vec{n}}. \quad (\text{A.3})$$

Therefore kinetic terms  $L_{\text{int}}$  describing interactions between vortices can be directly calculated from the Kähler potential as

$$\begin{aligned} \frac{g^2}{2\pi} L_{\text{int}} &= -\frac{1}{2} m_g^2 (\dot{r}^2 + r^2 \dot{\chi}^2) (c_e^2 K_0(m_e r) + c_g^2 (\vec{n}_1 \cdot \vec{n}_2) K_0(m_g r)) \\ &\quad + c_g^2 \left\{ \left( (\vec{n}_1 \cdot \dot{\vec{n}}_2) + (\vec{n}_2 \cdot \dot{\vec{n}}_1) \right) m_g \dot{r} + (\vec{n}_1 \times \vec{n}_2) \cdot (\dot{\vec{n}}_1 - \dot{\vec{n}}_2) m_g r \dot{\chi} \right\} K_1(m_g r) \\ &\quad - c_g^2 \left\{ -\vec{n}_1 \cdot \vec{n}_2 (|\dot{\vec{n}}_1|^2 + |\dot{\vec{n}}_2|^2) + \dot{\vec{n}}_1 \cdot \dot{\vec{n}}_2 + (\vec{n}_1 \times \dot{\vec{n}}_1) \cdot (\vec{n}_2 \times \dot{\vec{n}}_2) \right\} K_0(m_g r), \end{aligned} \quad (\text{A.4})$$

where  $z_1 - z_2 = z_{12} = r e^{i\chi}$ . The potential  $V_m$  induced by the mass deformation can be obtained by just replacing  $\dot{z}_I \rightarrow 0$  and  $\dot{\vec{n}}_I \rightarrow \vec{m} \times \vec{n}_I$  as

$$\begin{aligned} \frac{g^2}{2\pi} V_m &= \sum_I \frac{1}{2} |\vec{m} \times \vec{n}_I|^2 - c_g^2 \left\{ -\vec{n}_1 \cdot \vec{n}_2 (|\vec{m} \times \vec{n}_1|^2 + |\vec{m} \times \vec{n}_2|^2) \right. \\ &\quad \left. + (\vec{m} \times \vec{n}_1) \cdot (\vec{m} \times \vec{n}_2) + (\vec{n}_1 \times (\vec{m} \times \vec{n}_1)) \cdot (\vec{n}_2 \times (\vec{m} \times \vec{n}_2)) \right\} K_0(m_g r). \end{aligned} \quad (\text{A.5})$$

Due to the Lagrange multipliers, the equations of motion for the orientations become

$$(\mathbf{1}_3 - \vec{n}_I \vec{n}_I^T) \left( \frac{\partial \mathcal{L}}{\partial \vec{n}_I} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\vec{n}}_I} \right) = 0, \quad \Leftrightarrow \quad \vec{n}_I \times \left( \frac{\partial \mathcal{L}}{\partial \vec{n}_I} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\vec{n}}_I} \right) = 0. \quad (\text{A.6})$$

For instance, the equation of motion of the orientation  $\vec{n}$  for a single vortex is given by

$$\vec{n} \times \ddot{\vec{n}} + (\vec{m} \cdot \vec{n}) \vec{m} \times \vec{n} = 0, \quad (\text{A.7})$$

with the conservation conditions  $\vec{n} \cdot \dot{\vec{n}} = 0$ ,  $|\vec{n}|^2 = 1$ .

# References

- [1] N. S. Manton and P. Sutcliffe, “Topological solitons,” *Cambridge, UK: Univ. Pr. (2004)* 493 p
- [2] A. A. Belavin, A. M. Polyakov, A. S. Shvarts and Yu. S. Tyupkin, “Pseudoparticle solutions of the Yang-Mills equations,” *Phys. Lett. B* **59**, 85 (1975).
- [3] G. ’t Hooft, “Magnetic Monopoles In Unified Gauge Theories,” *Nucl. Phys. B* **79**, 276 (1974); A. M. Polyakov, “Particle spectrum in quantum field theory,” *JETP Lett.* **20**, 194 (1974) [*Pisma Zh. Eksp. Teor. Fiz.* **20**, 430 (1974)].
- [4] A. A. Abrikosov, “On the Magnetic properties of superconductors of the second group,” *Sov. Phys. JETP* **5**, 1174 (1957) [*Zh. Eksp. Teor. Fiz.* **32**, 1442 (1957)]; H. B. Nielsen and P. Olesen, “Vortex-line models for dual strings,” *Nucl. Phys. B* **61**, 45 (1973).
- [5] E. R. C. Abraham, P. K. Townsend, “Q kinks,” *Phys. Lett.* **B291**, 85-88 (1992); “More on Q kinks: A (1+1)-dimensional analog of dyons,” *Phys. Lett.* **B295**, 225-232 (1992); M. Arai, M. Naganuma, M. Nitta, N. Sakai, “Manifest supersymmetry for BPS walls in N=2 nonlinear sigma models,” *Nucl. Phys.* **B652**, 35-71 (2003). [[hep-th/0211103](#)]; “BPS wall in N=2 SUSY nonlinear sigma model with Eguchi-Hanson manifold,” In \*Arai, A. (ed.) et al.: A garden of quanta\* 299-325. [[hep-th/0302028](#)].
- [6] J. P. Gauntlett, D. Tong, P. K. Townsend, “Multidomain walls in massive supersymmetric sigma models,” *Phys. Rev.* **D64**, 025010 (2001). [[hep-th/0012178](#)]; D. Tong, “The Moduli space of BPS domain walls,” *Phys. Rev.* **D66**, 025013 (2002). [[hep-th/0202012](#)]; Y. Isozumi, K. Ohashi and N. Sakai, “Exact wall solutions in 5-dimensional SUSY QED at finite coupling,” *JHEP* **0311**, 060 (2003) [[arXiv:hep-th/0310189](#)]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai, Y. Tachikawa, “Global structure of moduli space for BPS walls,” *Phys. Rev.* **D71**, 105009 (2005). [[hep-th/0503033](#)].
- [7] E. B. Bogomolny, “Stability of Classical Solutions,” *Sov. J. Nucl. Phys.* **24**, 449 (1976); M. K. Prasad, C. M. Sommerfield, “An Exact Classical Solution for the ’t Hooft Monopole and the Julia-Zee Dyon,” *Phys. Rev. Lett.* **35**, 760-762 (1975).
- [8] E. Witten, D. I. Olive, “Supersymmetry Algebras That Include Topological Charges,” *Phys. Lett.* **B78**, 97 (1978).
- [9] N. S. Manton, “A Remark On The Scattering Of Bps Monopoles,” *Phys. Lett. B* **110** (1982) 54.



- [10] M. F. Atiyah and N. J. Hitchin, “Low-Energy Scattering Of Nonabelian Monopoles,” *Phys. Lett. A* **107**, 21 (1985); M. F. Atiyah and N. J. Hitchin, “The Geometry and Dynamics of Magnetic Monopoles. M.B. Porter Lectures,” *PRINCETON, USA: UNIV. PR. (1988) 133p*
- [11] G. W. Gibbons and N. S. Manton, “The Moduli space metric for well separated BPS monopoles,” *Phys. Lett. B* **356**, 32 (1995) [arXiv:hep-th/9506052].
- [12] K. M. Lee, E. J. Weinberg and P. Yi, “The Moduli Space of Many BPS Monopoles for Arbitrary Gauge Groups,” *Phys. Rev. D* **54**, 1633 (1996) [arXiv:hep-th/9602167].
- [13] C. H. Taubes, “Arbitrary N: Vortex Solutions To The First Order Landau-Ginzburg Equations,” *Commun. Math. Phys.* **72**, 277 (1980).
- [14] P. J. Ruback, “Vortex String Motion In The Abelian Higgs Model,” *Nucl. Phys.* **B296**, 669-678 (1988).
- [15] E. P. S. Shellard, P. J. Ruback, “Vortex Scattering In Two-dimensions,” *Phys. Lett.* **B209**, 262-270 (1988).
- [16] T. M. Samols, “Vortex Scattering,” *Commun. Math. Phys.* **145**, 149 (1992).
- [17] N. S. Manton and J. M. Speight, “Asymptotic interactions of critically coupled vortices,” *Commun. Math. Phys.* **236**, 535 (2003) [arXiv:hep-th/0205307].
- [18] H. Y. Chen and N. S. Manton, “The Kaehler potential of Abelian Higgs vortices,” *J. Math. Phys.* **46**, 052305 (2005) [arXiv:hep-th/0407011].
- [19] R. S. Ward, “Slowly Moving Lumps In The  $Cp^{*1}$  Model In (2+1)-Dimensions,” *Phys. Lett. B* **158**, 424 (1985).
- [20] M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi, N. Sakai, “Effective Action of Domain Wall Networks,” *Phys. Rev.* **D75**, 045010 (2007). [hep-th/0612003]; “Dynamics of Domain Wall Networks,” *Phys. Rev.* **D76**, 125025 (2007). [arXiv:0707.3267 [hep-th]].
- [21] M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi, N. Sakai, “Dynamics of Strings between Walls,” *Phys. Rev.* **D79**, 045015 (2009). [arXiv:0810.3495 [hep-th]].
- [22] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Webs of walls,” *Phys. Rev. D* **72**, 085004 (2005) [arXiv:hep-th/0506135]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Non-abelian webs of walls,” *Phys. Lett. B* **632**, 384 (2006) [arXiv:hep-th/0508241]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta and N. Sakai, “D-brane configurations for domain walls and their webs,” *AIP Conf. Proc.* **805**, 354 (2006) [arXiv:hep-th/0509127].

- [23] M. Eto, Y. Isozumi, M. Nitta and K. Ohashi, “ $1/2$ ,  $1/4$  and  $1/8$  BPS equations in SUSY Yang-Mills-Higgs systems: Field theoretical brane configurations,” Nucl. Phys. B **752**, 140 (2006) [arXiv:hep-th/0506257].
- [24] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Solitons in the Higgs phase: The moduli matrix approach,” J. Phys. A **39** (2006) R315 [arXiv:hep-th/0602170].
- [25] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “All exact solutions of a  $1/4$  Bogomol’nyi-Prasad-Sommerfield equation,” Phys. Rev. D **71**, 065018 (2005) [arXiv:hep-th/0405129].
- [26] E. Witten, “Some exact multipseudoparticle solutions of classical Yang-Mills theory,” Phys. Rev. Lett. **38**, 121 (1977).
- [27] S. Krusch and J. M. Speight, J. Math. Phys. **51**, 022304 (2010) [arXiv:0906.2007 [hep-th]].
- [28] E. J. Weinberg, “Multivortex Solutions Of The Ginzburg-landau Equations,” Phys. Rev. **D19**, 3008 (1979); “Index Calculations for the Fermion-Vortex System,” Phys. Rev. **D24**, 2669 (1981).
- [29] Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, “Construction of non-Abelian walls and their complete moduli space,” Phys. Rev. Lett. **93**, 161601 (2004). [hep-th/0404198]; “Non-Abelian walls in supersymmetric gauge theories,” Phys. Rev. **D70**, 125014 (2004). [hep-th/0405194]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai, “D-brane construction for non-Abelian walls,” Phys. Rev. **D71**, 125006 (2005). [hep-th/0412024]; A. Hanany, D. Tong, “On monopoles and domain walls,” Commun. Math. Phys. **266**, 647-663 (2006). [hep-th/0507140].
- [30] P. Goddard, J. Nuyts, D. I. Olive, “Gauge Theories and Magnetic Charge,” Nucl. Phys. **B125**, 1 (1977); C. Montonen, D. I. Olive, “Magnetic Monopoles as Gauge Particles?,” Phys. Lett. **B72**, 117 (1977); E. J. Weinberg, “Fundamental Monopoles and Multi-Monopole Solutions for Arbitrary Simple Gauge Groups,” Nucl. Phys. **B167**, 500 (1980); R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, H. Murayama, “NonAbelian monopoles,” Nucl. Phys. **B701**, 207-246 (2004). [hep-th/0405070]; M. Nitta and W. Vinci, Nucl. Phys. B **848**, 121 (2011) [arXiv:1012.4057 [hep-th]].
- [31] A. Hanany and D. Tong, “Vortices, instantons and branes,” JHEP **0307**, 037 (2003) [arXiv:hep-th/0306150]; R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, “Non-abelian superconductors: Vortices and confinement in  $N = 2$  SQCD,” Nucl. Phys. B **673**, 187 (2003) [arXiv:hep-th/0307287].

- [32] M. Shifman, A. Yung, “Localization of nonAbelian gauge fields on domain walls at weak coupling (D-brane prototypes II),” *Phys. Rev.* **D70**, 025013 (2004); [hep-th/0312257]; M. Eto, M. Nitta, K. Ohashi, D. Tong, “Skyrmions from instantons inside domain walls,” *Phys. Rev. Lett.* **95**, 252003 (2005). [hep-th/0508130]; M. Eto, T. Fujimori, M. Nitta, K. Ohashi, N. Sakai, “Domain walls with non-Abelian clouds,” *Phys. Rev.* **D77**, 125008 (2008). [arXiv:0802.3135 [hep-th]].
- [33] D. Tong, “TASI lectures on solitons,” arXiv:hep-th/0509216; “Quantum Vortex Strings: A Review,” *Annals Phys.* **324**, 30 (2009) [arXiv:0809.5060 [hep-th]]; K. Konishi, “The magnetic monopoles seventy-five years later,” *Lect. Notes Phys.* **737**, 471 (2008) [arXiv:hep-th/0702102]; “Advent of Non-Abelian Vortices and Monopoles— further thoughts about duality and confinement,” *Prog. Theor. Phys. Suppl.* **177**, 83 (2009) [arXiv:0809.1370 [hep-th]]; M. Shifman and A. Yung, “Supersymmetric Solitons and How They Help Us Understand Non-Abelian Gauge Theories,” *Rev. Mod. Phys.* **79**, 1139 (2007) [arXiv:hep-th/0703267]; an expanded version in Cambridge University Press, 2009.
- [34] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Moduli space of non-Abelian vortices,” *Phys. Rev. Lett.* **96** (2006) 161601 [arXiv:hep-th/0511088].
- [35] M. Eto, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci and N. Yokoi, “Non-Abelian vortices of higher winding numbers,” *Phys. Rev. D* **74** (2006) 065021 [arXiv:hep-th/0607070].
- [36] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Manifestly supersymmetric effective Lagrangians on BPS solitons,” *Phys. Rev. D* **73**, 125008 (2006) [arXiv:hep-th/0602289].
- [37] M. Eto *et al.*, “Non-Abelian duality from vortex moduli: a dual model of color-confinement,” *Nucl. Phys. B* **780**, 161 (2007) [arXiv:hep-th/0611313].
- [38] M. Eto, T. Fujimori, S. Bjarke Gudnason, Y. Jiang, K. Konishi, M. Nitta, K. Ohashi, “Group Theory of Non-Abelian Vortices,” *JHEP* **1011**, 042 (2010). [arXiv:1009.4794 [hep-th]].
- [39] M. Eto, K. Hashimoto, G. Marmorini, M. Nitta, K. Ohashi and W. Vinci, “Universal reconnection of non-Abelian cosmic strings,” *Phys. Rev. Lett.* **98** (2007) 091602 [arXiv:hep-th/0609214].
- [40] T. Fujimori, G. Marmorini, M. Nitta, K. Ohashi, N. Sakai, “The Moduli Space Metric for Well-Separated Non-Abelian Vortices,” *Phys. Rev.* **D82**, 065005 (2010). [arXiv:1002.4580 [hep-th]].

- [41] B. Collie, “Dyonic Non-Abelian Vortices,” J. Phys. A **42**, 085404 (2009) [arXiv:0809.0394 [hep-th]].
- [42] M. Shifman and A. Yung, “Non-Abelian string junctions as confined monopoles,” Phys. Rev. D **70**, 045004 (2004) [arXiv:hep-th/0403149]; A. Hanany and D. Tong, “Vortex strings and four-dimensional gauge dynamics,” JHEP **0404**, 066 (2004) [arXiv:hep-th/0403158].
- [43] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Instantons in the Higgs phase,” Phys. Rev. D **72**, 025011 (2005) [arXiv:hep-th/0412048]; T. Fujimori, M. Nitta, K. Ohta, N. Sakai, M. Yamazaki, “Intersecting Solitons, Amoeba and Tropical Geometry,” Phys. Rev. D **78**, 105004 (2008). [arXiv:0805.1194 [hep-th]].
- [44] M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi and N. Sakai, “Multiple Layer Structure of Non-Abelian Vortex,” Phys. Lett. B **678**, 254 (2009) [arXiv:0903.1518 [hep-th]].
- [45] T. Vachaspati and A. Achucarro, “Semilocal cosmic strings,” Phys. Rev. D **44**, 3067 (1991); A. Achucarro and T. Vachaspati, “Semilocal and electroweak strings,” Phys. Rept. **327**, 347 (2000) [Phys. Rept. **327**, 427 (2000)] [arXiv:hep-ph/9904229].
- [46] R. A. Leese and T. M. Samols, “Interaction of semilocal vortices,” Nucl. Phys. B **396**, 639 (1993).
- [47] R. A. Leese, “Q lumps and their interactions,” Nucl. Phys. B **366**, 283 (1991).
- [48] M. Eto, T. Fujimori, M. Nitta, K. Ohashi, K. Ohta and N. Sakai, “Statistical Mechanics of Vortices from D-branes and T-duality,” Nucl. Phys. B **788**, 120 (2008) [arXiv:hep-th/0703197].
- [49] M. Shifman and A. Yung, “Non-Abelian semilocal strings in  $N = 2$  supersymmetric QCD,” Phys. Rev. D **73**, 125012 (2006) [arXiv:hep-th/0603134].
- [50] M. Eto *et al.*, “On the moduli space of semilocal strings and lumps,” Phys. Rev. D **76** (2007) 105002 [arXiv:0704.2218 [hep-th]].
- [51] M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, M. Nitta, K. Ohashi, W. Vinci, “Constructing Non-Abelian Vortices with Arbitrary Gauge Groups,” Phys. Lett. **B669**, 98-101 (2008). [arXiv:0802.1020 [hep-th]].
- [52] M. Eto, T. Fujimori, S. B. Gudnason, M. Nitta, K. Ohashi, “SO and USp Kahler and Hyper-Kahler Quotients and Lumps,” Nucl. Phys. **B815**, 495-538 (2009). [arXiv:0809.2014 [hep-th]]; M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, T. Nagashima, M. Nitta, K. Ohashi,

- W. Vinci, “Non-Abelian Vortices in  $SO(N)$  and  $USp(N)$  Gauge Theories,” JHEP **0906**, 004 (2009). [arXiv:0903.4471 [hep-th]].
- [53] L. Ferretti, S. B. Gudnason, K. Konishi, “Non-Abelian vortices and monopoles in  $SO(N)$  theories,” Nucl. Phys. **B789**, 84-110 (2008). [arXiv:0706.3854 [hep-th]]; S. B. Gudnason, K. Konishi, “Low-energy  $U(1) \times USp(2M)$  gauge theory from simple high-energy gauge group,” Phys. Rev. **D81**, 105007 (2010). [arXiv:1002.0850 [hep-th]]; S. B. Gudnason, Y. Jiang, K. Konishi, “Non-Abelian vortex dynamics: Effective world-sheet action,” JHEP **1008**, 012 (2010). [arXiv:1007.2116 [hep-th]].
- [54] M. Eto, T. Fujimori, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta and N. Sakai, “Non-Abelian vortices on cylinder: Duality between vortices and walls,” Phys. Rev. D **73**, 085008 (2006) [arXiv:hep-th/0601181].
- [55] J. M. Baptista, “Non-abelian vortices on compact Riemann surfaces,” Commun. Math. Phys. **291**, 799 (2009) [arXiv:0810.3220 [hep-th]]; A. D. Popov, “Integrability of Vortex Equations on Riemann Surfaces,” Nucl. Phys. B **821**, 452 (2009) [arXiv:0712.1756 [hep-th]]; “Non-Abelian Vortices on Riemann Surfaces: an Integrable Case,” Lett. Math. Phys. **84**, 139 (2008) [arXiv:0801.0808 [hep-th]].
- [56] J. M. Baptista, “On the  $L^2$ -metric of vortex moduli spaces,” Nucl. Phys. **B844**, 308-333 (2011). [arXiv:1003.1296 [hep-th]].
- [57] J. M. Speight, “Static intervortex forces,” Phys. Rev. D **55**, 3830 (1997) [arXiv:hep-th/9603155].
- [58] R. Auzzi, M. Eto and W. Vinci, “Static Interactions of non-Abelian Vortices,” JHEP **0802**, 100 (2008) [arXiv:0711.0116 [hep-th]]; R. Auzzi, M. Eto, S. B. Gudnason, K. Konishi and W. Vinci, “On the Stability of Non-Abelian Semi-local Vortices,” Nucl. Phys. B **813**, 484 (2009) [arXiv:0810.5679 [hep-th]].
- [59] B. Collie and D. Tong, “The Dynamics of Chern-Simons Vortices,” Phys. Rev. D **78**, 065013 (2008) [arXiv:0805.0602 [hep-th]]; S. B. Gudnason, “Non-Abelian Chern-Simons vortices with generic gauge groups,” Nucl. Phys. B **821**, 151 (2009) [arXiv:0906.0021 [hep-th]].
- [60] K. Hashimoto, D. Tong, “Reconnection of non-Abelian cosmic strings,” JCAP **0509**, 004 (2005). [hep-th/0506022].